Math 280Y: Arithmetic Statistics

Spring 2023

Problem set #5

due Friday, April 7 at 10pm

Problem 1. Identify the space V_n of monic polynomials of degree n with $\mathbb{R}^n$ by sending $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in \mathbb{R}[X]$ to $(a_{n-1}, \ldots, a_0)$. Let $S \subseteq V_3 = \mathbb{R}^3$ be the set of polynomials that have two complex roots and one real root, and such that all roots have absolute value at most 1. Compute the volume of S. (Use a computer if you like.)

Problem 2 (bonus)**.** Let K be a number field and let $1 = \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ be the successive minima of $\mathcal{O}_K$.

a) Show that $\lambda_n \ll \lambda_i \lambda_{n+1-i}$ for all $1 \leqslant i \leqslant n$.

b) Show that $\lambda_n \ll |\operatorname{disc}(K)|^{1/n}$.

Problem 3 (Theorem 12.2)**.** In class, we defined two maps between the sets

$$\{K\text{-algebra } L \text{ of degree } n \text{ (up to isomorphism)}\}$$

and

$$S_n \backslash \operatorname{Hom}_{\mathrm{cont}}(\Gamma_K, S_n).$$

Show that they are inverses.

Problem 4 (Theorem 12.3)**.** Show that if L corresponds to a continuous homomorphism $f : \Gamma_K \to S_n$, then $\operatorname{Aut}(L) \cong \operatorname{Stab}_{S_n}(f)$.

Problem 5. Let K be a nonarchimedean local field with prime ideal $\mathfrak{p}$ and residue field $\mathbb{F}_q$.

a) Show that $\int_{\mathcal{O}_K} |x| \mathrm{d}x = 1 - \frac{1}{q+1}$.

b) Let $f(X) \in \mathcal{O}_K[X]$ be a polynomial such that $f'(X) \bmod \mathfrak{p}$ has k simple roots in $\mathbb{F}_q$ and no roots of higher multiplicity in $\mathbb{F}_q$. For any $y \in \mathcal{O}_K$, let $m(y)$ be the number of $x \in \mathcal{O}_K$ such that $f(x) = y$. Show that

$$\int_{\mathcal{O}_K} m(y) \mathrm{d}y = 1 - \frac{k}{q+1}.$$

(This is the expected number of preimages of a random element $y \in \mathcal{O}_K$ under the map $f : \mathcal{O}_K \to \mathcal{O}_K$.)