Math 280Y: Arithmetic Statistics

Spring 2023

Problem set #4

due Sunday, March 12 at 10pm

Problem 1. Let $F_d(K)$ be the $(n+1)$-dimensional vector space of degree n forms $f \in K[X,Y]$. We define an action of $\mathrm{GL}_2(K)$ on $F_d(K)$ as follows:

$$(Mf)(v) = f(M^T v) \qquad \text{for } M \in \mathrm{GL}_2(K) \text{ and } f \in F_d(K) \text{ and } v \in K^2.$$

(Note that this action on $F_2(K)$ is off by a factor of $\det(M)$ from the action on $\mathcal{V}(K)$ defined in class!)

a) Show that the linear map $F_d(K) \to F_d(K)$ given by $f \mapsto Mf$ has determinant $\det(M)^{n+1}$.
Hint: It suffices to consider elementary matrices M.

b) Show that $\mathrm{disc}(Mf) = \det(M)^{2(n-1)} \cdot \mathrm{disc}(f)$.

Problem 2. Let K be a nonarchimedean local field of characteristic zero and let $L = K(\sqrt{D})$ be a quadratic field extension. Assume that $\mathcal{O}_L = \mathcal{O}_K[\frac{D+\sqrt{D}}{2}]$. Show that the action of $\mathrm{GL}_2(\mathcal{O}_K)$ on $\{f \in V(\mathcal{O}_K) \mid \mathrm{disc}(f) = D\}$ is transitive.

Problem 3. An element $x \neq 0$ of a number field L is *totally positive* if $\sigma(x) > 0$ for all real embeddings $\sigma : L \to \mathbb{R}$. The *narrow class group* of L is the group of fractional ideals modulo the group of totally positive elements. Construct a bijection between the narrow class group of a quadratic number field L with discriminant D and the set $\mathrm{SL}_2(\mathbb{Z})\backslash\{f \in \mathcal{V}(\mathbb{Z}) \mid \mathrm{disc}(f) = D\}$.

Problem 4. Let K be a quadratic number field of discriminant D. In class, we've constructed a bijection

$$\mathrm{Cl}_K = K^\times\backslash\{I \text{ fractional ideal of } K\} \longleftrightarrow \mathrm{GL}_2(\mathbb{Z})\backslash\mathcal{V}_{\mathrm{disc}=D}(\mathbb{Z}).$$

Let $\mathcal{W}(\mathbb{Z}) = \mathcal{V}(\mathbb{Z}) \times \mathbb{Z}^2$ be the set of pairs $e = (f, v)$, where f is a binary quadratic form with integer coefficients, and $v \in \mathbb{Z}^2$. Let $\mathrm{disc}(e) = \mathrm{disc}(f)$ and $\mathrm{Nm}(e) = f(v)$. Furthermore, let $\mathrm{GL}_2(\mathbb{Z})$ act on $\mathcal{W}(\mathbb{Z})$ by $M.(f, v) = (M.f, \det(M)(M^T)^{-1}v)$ (where the action on $\mathcal{V}(\mathbb{Z})$ was defined in class by $(M.f)(w) = f(M^T w)/\det(M)$). For any $N \geqslant 1$, let $\mathcal{W}_{\mathrm{disc}=D, |\mathrm{Nm}|=N} \subset \mathcal{W}$ be the set of $e \in \mathcal{W}$ with $\mathrm{disc}(e) = D$ and $|\mathrm{Nm}(e)| = N$.

a) Construct a bijection

$$\{I \subseteq \mathcal{O}_K \text{ ideal of } \mathcal{O}_K \mid \operatorname{Nm}(I) = N\} \longleftrightarrow \operatorname{GL}_2(\mathbb{Z})\backslash \mathcal{W}_{\mathrm{disc}=D, |\operatorname{Nm}|=N}(\mathbb{Z}).$$

b) What is the $\operatorname{GL}_2(\mathbb{Z})$-stabilizer of an element of $\mathcal{W}_{\mathrm{disc}=D, |\operatorname{Nm}|=N}(\mathbb{Z})$?