Math 280Y: Arithmetic Statistics

Spring 2023

Problem set #2

due Friday, February 10 at 10pm

Problem 1. For any t, the discriminant of the polynomial $f_t(X) = X^3 - tX^2 + (t-3)X + 1$ is a square: $\operatorname{disc}(f_t) = (9 - 3t + t^2)^2$. Assuming the discriminant is nonzero (the polynomial $f_t(X)$ is squarefree), this implies that either $f_t(X)$ splits into linear factors, or its Galois group is the cyclic group $A_3 \subset S_3$ of degree three.

a) Show that

$$\lim_{q\to\infty} \mathbb{P}_{t\in\mathbb{F}_q}(f_t(X) \text{ splits into linear factors over } \mathbb{F}_q) = \mathbb{P}_{\pi\in A_3}(\pi = \mathrm{id}) = \frac{1}{3}.$$

b) Show that

$$\mathbb{P}_{t\in\mathbb{Z}}(f_t(X) \text{ splits into linear factors over } \mathbb{Q}) = 0.$$

Problem 2. a) Let $(v_1, \ldots, v_n)$ be a directional basis for a full lattice Λ in $\mathbb{R}^n$. Show that the lattice Λ' spanned by $v_1, \ldots, v_n$ has index at most $n!$ in Λ.

b) Let $\lambda_1, \ldots, \lambda_n$ be the successive minima of a full lattice Λ in $\mathbb{R}^n$. Show that the successive minima $\lambda'_1, \ldots, \lambda'_n$ of the dual lattice $\Lambda' = \{x \in \mathbb{R}^n \mid x \cdot y \in \mathbb{Z} \text{ for all } y \in \Lambda\}$ satisfy $\lambda'_i \asymp_{n,\|\cdot\|} \lambda^{-1}_{n+1-i}$ for $i = 1, \ldots, n$.

Problem 3. a) Let Λ be a full lattice in $\mathbb{R}^2$ and consider any norm $\|\cdot\|$ on $\mathbb{R}^2$. Let the successive minima be $\lambda_1 \leqslant \lambda_2$. Show that the lattice Λ is spanned by some directional basis (v_1, v_2) for Λ.
Hint: Use *Pick's theorem.*

b) Prove or disprove: Every directional basis for Λ spans Λ.

c) Let K be the smallest centrally symmetric convex subset of $\mathbb{R}^3$ that contains $(1,0,0)$, $(0,1,0)$, and $(1,1,2)$. Show that the vectors in $K \cap \mathbb{Z}^3$ span $\mathbb{R}^3$ but not $\mathbb{Z}^3$.

Problem 4. For any function f on $\mathbb{R}^n$ and any $R > 0$, we let $f_R(x) = f(x/R)$. Let $f \in L^1(\mathbb{R}^n)$ be an integrable function and let $g : \mathbb{R}^n \to \mathbb{C}$ be a smooth compactly supported function. Show that for all $\varepsilon > 0$ and $k \geqslant 0$, as $R \to \infty$, we have

$$\sum_{x \in \mathbb{Z}^n} (f_R * g_{R^\varepsilon})(x) = R^{(1+\varepsilon)n} \cdot \int_{\mathbb{R}^n} f(x)\mathrm{d}x \cdot \int_{\mathbb{R}^n} g(x)\mathrm{d}x + \mathcal{O}_{f,g,\varepsilon,k}(R^{-k}).$$

Problem 5 (bonus)**.** Show that the following claim is false: For every compact subset A of $\mathbb{R}$, as $R \to \infty$, we have:

$$\#((R \cdot A) \cap \mathbb{Z}) = R \cdot \mathrm{vol}(A) + \mathcal{O}_A(1).$$

(This contradicts the first paragraph of Davenport's paper "On a principle of Lipschitz".)