Let $\mathrm{ram}(f) := \prod_{\substack{p: \\ f \text{ ramified at } p}} p$ [prod. of ramified primes].

Question Let $K = \mathbb{Q}$.

$$N(T) := \#\{f : \Gamma_K^{ab} \to G \mid \mathrm{ram}(f) \le T\} \sim ? \text{ as } T \to \infty$$

Approach #1: Use Kummer theory over an extension $\mathbb{Q}(\zeta_n)$ of $\mathbb{Q}$ with $nG = \{id\}$.
Think about which hom. $\Gamma_{\mathbb{Q}(\zeta_n)} \to G$ extend to hom. $\Gamma_{\mathbb{Q}} \to G$, and in how many ways.

Approach #2: Class field theory

Thm 16.3 (Kronecker-Weber)

a) $\mathbb{Q}^{ab} = \bigcup_{n \ge 1} \mathbb{Q}(\zeta_n) = \bigcup_{p} \bigcup_{k \ge 1} \mathbb{Q}(\zeta_{p^k})$

b) $\Gamma_{\mathbb{Q}}^{ab} \cong \varprojlim_{n \ge 1} (\mathbb{Z}/n\mathbb{Z})^\times \cong \prod_p \varprojlim_{k \ge 1} (\mathbb{Z}/p^k\mathbb{Z})^\times = \prod_p \mathbb{Z}_p^\times$

c) $I_p = \mathbb{Z}_p^\times \subseteq \prod_{p'} \mathbb{Z}_{p'}^\times$

Lemma 16.4 Assume $p \nmid \#G$. Let $\mathcal{C}$ ~~be~~ the set of nontrivial cyclic subgroups of G.

Then, $$\#\{f : \mathbb{Z}_p^\times \to G \text{ (cont.)}\} = \sum_{\substack{U \in \mathcal{C}: \\ p \equiv 1 \bmod \#U}} \varphi(\#U)$$

Euler's totient function

Pf f must factor through $(\mathbb{Z}/p^k\mathbb{Z})^\times$ for some $k \geq 1$.

We have $x^{p^{k-1}} \equiv 1 \bmod p^k$ for any $x \equiv 1 \bmod p$.

$\Rightarrow f(x^{p^{k-1}}) = id$ for all $x \equiv 1 \bmod p$.

$$f(x^{p^{k-1}}) = f(x)^{p^{k-1}}$$

$\underset{p \nmid \#G}{\Longrightarrow}$ $f(x) = id$ — " —

$\Rightarrow f$ factors through $(\mathbb{Z}/p\mathbb{Z})^\times \cong C_{p-1}$.

[cyclic grp. of order $p-1$]

Let $U = \mathrm{im}(f)$.

$\Rightarrow p \equiv 1 \bmod \#U$.

In this case, there is exactly one factor group of C_{p-1} isom. to U, with $\varphi(\#U)$ isomorphisms.

$\square$

Let $D_p(s) := \sum_{f:\mathbb{Z}_p^\times \to G} \underbrace{\mathrm{ram}_p(f)^{-s}}_{\substack{1 \text{ if } f \text{ is trivial map} \\ p^{-s} \text{ otherwise}}} = 1 + c_p \cdot p^{-s}$

~~We've shown~~

Remarks

a) This is a finite sum.

b) We've shown $c_p = \sum_{U \in \dots :\, p \equiv 1 \bmod \#U} \varphi(\#U)$ if $p \nmid \#G$. (HW!)

Since $\Gamma_{\mathbb{Q}}^{ab} = \prod \mathbb{Z}_p^\times$, we have

$$D(s) := \sum_{f:\Gamma_{\mathbb{Q}}^{ab} \to G} \mathrm{ram}(f)^{-s} = \prod_p D_p(s). \qquad \text{(formally)}$$

Write $D(s) = \sum_{n \geq 1} a_n n^{-s}$ (formally).

~~By Wiener Ikehara~~

~~Using Ikehara Theorem~~

By the Wiener-Ikehara Theorem / Kato's Thm / Perron's formula / Tauberian Th, the asymptotics of $\sum_{n \in T} a_n = \#\{f:\Gamma_{\mathbb{Q}}^{ab} \to G \mid \mathrm{ram}(f) \in T\}$ are determined by the rightmost pole of $D(s)$.

~~Write $D_1(s) \sim D_2(s)$ if $\frac{D_1(s)}{D_2(s)}$ can be holomorphically continued to~~

Thm 16.5

$D(s)$ can be meromorphically continued to $\{\mathrm{Re}(s) \geq 1\}$, holomorphic except for a pole of order $\#\mathcal{C}$ at $s=1$.

Pf Write $D_1(s) \sim D_2(s)$ if $\frac{D_1(s)}{D_2(s)}$ can be holomorphically continued to $\{\mathrm{Re}(s) \geq 1\}$. Write $A(s) \approx B(s)$ if $\frac{A(s)}{B(s)} = 1 + \mathcal{O}(p^{-2s})$ and $\frac{B(s)}{A(s)} = 1 + \mathcal{O}(p^{-2s})$.

For any $p \nmid \#G$, we have

$$D_p(s) = 1 + \sum_{\substack{U \in \mathcal{C}: \\ p \equiv 1 \bmod \#U}} \varphi(\#U) p^{-s} \approx \prod_{\substack{U \in \mathcal{C} \\ p \equiv 1 \bmod \#U}} \left(1 + \varphi(\#U) p^{-s}\right)$$

$$= \prod_{U \in \mathcal{C}} \Big(1 + \sum_{\substack{\chi: (\mathbb{Z}/|U|\mathbb{Z})^\times \to \mathbb{C}^\times \\ \text{group hom.}}} \chi(p) \cdot p^{-s}\Big)$$

$$\approx \prod_{U \in \mathcal{C}} \prod_{\chi} \left(1 + \chi(p) \cdot p^{-s}\right)$$

$$\approx \prod_{U \in \mathcal{C}} \prod_{\chi} \left(1 + \chi(p) \cdot p^{-s} + \chi(p^2) \cdot p^{-2s} + \ldots\right).$$

$$\Rightarrow D(s) = \prod_p D_p(s) \sim \prod_{U \in \mathcal{C}} \prod_{\chi} L(s, \chi).$$

~~The RHS is hol. on $\{\mathrm{Re}(s)\geq 1\}$ except~~

The Dirichlet L-series $L(s,\chi)$ is hol. on $\{\mathrm{Re}(s)\geq 1\}$ except for a simple pole at $s=1$ in case χ is the trivial hom. $(\mathbb{Z}/|U|\mathbb{Z})^\times \to \mathbb{C}^\times$. $\square$

Using some form of Wiener-Ikehara, it follows that:

Thm 16.6 There is a constant $C>0$ such that
$$N(T) \sim C\cdot T(\log T)^{|\mathcal{C}|-1} \quad \text{for } T\to\infty.$$

Instead of combining all ramified primes in the same invariant, one can for each $U\in\mathcal{C}$ define

$$\mathrm{inv}_U(f) := \prod_{\substack{p\nmid \#G:\\ f(\underbrace{I(p)}_{\mathbb{Z}_p^\times})=U}} p \qquad \text{for } f:\underbrace{\Gamma_{\mathbb{Q}}^{ab}}_{\prod_p \mathbb{Z}_p^\times}\to G.$$

Rmk $\displaystyle\prod_U \mathrm{inv}_U(f) = \prod_{\substack{p\nmid\#G\\ p\mid \mathrm{ram}(f)}} p$.

Using similar techniques as above (e.g. Wiener-Ikehara for Dirichlet series in $|\mathcal{C}|$ variables), one can show:

Prop 16.7 $\#\{f \mid \mathrm{inv}_U(f)\leq T_U\ \forall U\in\mathcal{C}\} \sim C'\cdot\prod_U T_U$ for all $T_U\to\infty$.

Rmk 16.8 b) Thm 16.6 can be recovered from ~~[illegible]~~ a).

a) The same still holds (with a different constant) if we fix the restriction of f to $\mathbb{Z}_p^\times$ for finitely many p or if we fix some of the invariants $\mathrm{inv}_v(f) = i_v$ instead of $\mathrm{inv}_v(f) \leq T_v$ with $T_v \to \infty$.

c) You can use this to count e.g. Galois extensions $L/\mathbb{Q}$ with group G by $\mathrm{ram}(L)$ or $\mathrm{disc}(L)$ or...

Previously done by Wright: Distribution of discriminants of abelian ext.