Step 1: construct a fund. dom.$^{\alpha_T}$ for $GL_2(\mathbb{Z}) \circlearrowright \{f \in V(\mathbb{R}) \mid 0 < |\mathrm{disc}(f)| \in T\}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad V_T(\mathbb{R})$

Idea: $GL_2(\mathbb{Z}) \circlearrowright GL_2^{\pm 1}(\mathbb{R}) \circlearrowright V_T(\mathbb{R})$

$$GL_2^{\pm 1}(\mathbb{R}) = \{g \in GL_2(\mathbb{R}) \mid \det(g) = \pm 1\}$$

Let σ be a fund. dom. for the action of $GL_2(\mathbb{Z})$ on $GL_2^{\pm 1}(\mathbb{R})$ by left mult.

Let β_T be a fund. dom. for $GL_2^{\pm}(\mathbb{R}) \circlearrowright V_T(\mathbb{R})$.

Let

$$\alpha_T(f) := \sum_{\substack{g \in GL_2^{\pm 1}(\mathbb{R}),\\ f' \in V_T(\mathbb{R}):\\ f = gf'}} \sigma(g)\,\beta_T(f').$$

Rmk 15.5.7
α_T is a fund. dom. for $GL_2(\mathbb{Z}) \circlearrowright V_T(\mathbb{R})$.

Pf

$$\sum_{h \in GL_2(\mathbb{Z})} \alpha_T(hf) = \sum_{h \in GL_2(\mathbb{Z})} \sum_{\substack{g \in GL_2^{\pm 1}(\mathbb{R}),\\ f' \in V_T(\mathbb{R}):\\ hf = gf'}} \sigma(g)\,\beta_T(f')$$

$$= \sum_{\substack{g \in GL_2^{\pm 1}(\mathbb{R}),\\ f' \in V_T(\mathbb{R}):\\ f = gf'}} \underbrace{\sum_{h \in GL_2(\mathbb{Z})} \sigma(hg)}_{1}\,\beta_T(f')$$

$$= \sum_{g \in GL_2^{\pm 1}(\mathbb{R})} \beta_T(g^{-1}f) = 1$$

$\square$

To construct β_T:

$$GL_2(\mathbb{R})\backslash\{f\in V(\mathbb{R}) \mid \mathrm{disc}\neq 0\} \longleftrightarrow \{\text{étale cubic ext.}^{\text{s}} \text{ of } \mathbb{R}\}/\cong$$

$$\{\text{étale cubic ext.}^{\text{s}} \text{ of } \mathbb{R}\}/\cong \;=\; \{\mathbb{R}\times\mathbb{R}\times\mathbb{R},\ \mathbb{R}\times\mathbb{C}\}$$

$\begin{pmatrix}\lambda & \\ & \lambda\end{pmatrix} f = \lambda\cdot f$ $\rightarrow \updownarrow$

$$GL_2^{\pm1}(\mathbb{R})\backslash\{f\in V(\mathbb{R}) \mid |\mathrm{disc}|=1\}$$

$$\mathrm{Stab}_{GL_2^{\pm1}(\mathbb{R})}(f) = \mathrm{Stab}_{GL_2(\mathbb{R})}(f) \cong \mathrm{Aut}_{\mathbb{R}}(L)$$

$\uparrow$ $\mathrm{disc}(gf) = \det(g)^2\,\mathrm{disc}(f)$

$\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ corr. to $f_1 = -X^2Y+XY^2$ with $\#\mathrm{Stab} = \#\mathrm{Aut} = 6 =: r_1$

$\mathbb{R}\times\mathbb{C}$ corr. to $f_2 = \frac{1}{\sqrt{2}}X(X^2+Y^2)$ with $\#\mathrm{Stab} = \#\mathrm{Aut} = 2 =$

$$\Rightarrow \quad \beta_T(f) := \begin{cases} 1/6, & f = \lambda\cdot f_1 \text{ for some } 0<\lambda\leq T^{1/4} \\ 1/2, & f = \lambda\cdot f_2 \text{ for some } 0<\lambda\leq T^{1/4} \\ 0, & \text{otherwise} \end{cases}$$

is a fund. dom. for $GL_2^{\pm1}(\mathbb{R}) \circlearrowright V_T(\mathbb{R})$.

Step 2: Compute $V \cdot T := \int_{V(\mathbb{R})} \alpha_T(f)\, df$.

For $i = 1, 2$, consider the map

$$\psi_i : \mathbb{R}_{>0} \times GL_2^{\pm 1}(\mathbb{R}) \longrightarrow V(\mathbb{R})$$
$$(\lambda, g) \longmapsto \lambda \cdot g f_i$$

We have $\alpha_T(f) = \sum_{i=1}^{2} \sum_{\substack{(\lambda, g): \\ \psi_i(\lambda, g) = f}} \frac{1}{r_i} \cdot \mathbb{1}_{(0, T^{1/4}]}(\lambda) \cdot \sigma(g)$.

$$\Longrightarrow \int_{V(\mathbb{R})} \alpha_T(f) = \sum_{i=1}^{2} \frac{1}{r_i} \cdot \int_{\mathbb{R}_{>0}} \mathbb{1}_{(0, T^{1/4}]}(\lambda) \cdot 2 \lambda^4 d^\times \lambda \cdot \int_{GL_2^{\pm 1}(\mathbb{R})} \sigma(g)\, d^\times g$$

(↑ change of variables (using Lemma 15.1.2 c and section 7.1) & ~~Note:~~ Both sides are 4-dimensional)

$$= \left(\sum_{\substack{L \text{ étale} \\ \mathbb{R}\text{-alg.} \\ \text{of deg. } 3}} \frac{1}{\# \mathrm{Aut}(L)} \right) \cdot \frac{1}{2} T \cdot \underbrace{\mathrm{vol}(GL_2(\mathbb{Z}) \backslash GL_2^{\pm 1}(\mathbb{R}))}_{= \mathrm{vol}(SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R}))}$$

$$\Longrightarrow \boxed{V = \left(\sum_{L/\mathbb{R}} \frac{1}{\# \mathrm{Aut}(L)} \right) \cdot \frac{1}{2} \cdot \mathrm{vol}(SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R}))}$$

$$\boxed{= \frac{1}{3} \cdot \mathrm{vol}(SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R}))}$$

<u>Step 3</u> Show $\sum_{f \in V^{a\neq 0}(\mathbb{Z}) \cap \Lambda} \alpha_T(f) \underset{\Lambda}{\sim} \frac{V}{\mathrm{covol}(\Lambda)} \cdot T$ for $T \to \infty$.

Let σ_0 be Siegel's fund. dom. for $GL_2(\mathbb{Z}) \circlearrowright GL_2^{\pm 1}(\mathbb{R})$:

$\mathrm{supp}(\sigma_0) \subseteq N' A' O_n(\mathbb{R})$, where $N' = \left\{ \begin{pmatrix} 1 & 0 \\ n_{21} & 1 \end{pmatrix} \middle| \, |n_{21}| \leq \frac{1}{2} \right\}$

and $A' = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \middle| \begin{array}{l} a_1, a_2 > 0 \\ a_2 \geq \frac{\sqrt{3}}{2} a_1 \end{array} \right\}$

Let $\eta : GL_2^{\pm 1}(\mathbb{R}) \to \mathbb{R}_{\geq 0}$ be smooth and compactly supported, or the indicator fct. of a (cpt.) semialg. set, with $\int \eta(g)\, dg = 1$.

Then, $\sigma := \sigma_0 * \eta$ is a fund. dom. for $GL_2(\mathbb{Z}) \circlearrowright GL_2^{\pm 1}(\mathbb{R})$ by Lemma 5.4.

Let $\gamma_T(f) := \sum_{\substack{g \in GL_2^{\pm 1}(\mathbb{R}), \\ f' \in V_T(\mathbb{R}): \\ f = gf'}} \eta(g) \beta_T(f')$.

Then, $\alpha_T(f) = \int_{GL_2^{\pm 1}(\mathbb{R})} \sigma_0(g) \gamma_T(g^{-1} f)$ by the def. of $\sigma = \sigma_0 * \eta$.

$$\Rightarrow \sum_{f \in V^{a\neq 0}(\mathbb{Z}) \cap \Lambda} \alpha_T(f) = \int_{GL_2^{\pm 1}(\mathbb{R})} \sigma_0(g) \underbrace{\sum_{f \in V^{a\neq 0}(\mathbb{Z}) \cap \Lambda} \gamma_T(g^{-1} f)}_{g\gamma_T(f)} \, dg \qquad (I)$$

Let $g = \begin{pmatrix}1 & 0\\ n & 1\end{pmatrix}\begin{pmatrix}a_1 & 0\\ 0 & a_2\end{pmatrix}k \in N'A'O_n(\mathbb{R}) \cap GL_2^{\pm 1}(\mathbb{R})$.

~~We have $\mathrm{supp}(g\gamma_T) \subseteq \{f = ax^3 + \dots$ [illegible] $\mid a \ll$~~

If $f \in \underbrace{\mathrm{supp}(\gamma_T)}_{\substack{\text{compact,}\\ \text{contained in}\\ \text{box of size} \asymp T^{1/4}}}$, then $gf(1,0) = f(\underbrace{g^T\binom{1}{0}}_{\|\cdot\| \ll a_1}) \ll a_1^3 T^{1/4}$.

$\Rightarrow \mathrm{supp}(g\gamma_T) \subseteq \{f = ax^3 + \dots \mid a \ll a_1^3 T^{1/4}\}$.

$\Rightarrow$ If $a_1^3 T^{1/4} \ll 1$, then $\sum_{f \in V^{a\neq 0}(\mathbb{Z})} g\gamma_T(f) = 0$. (II)

Otherwise, apply Poisson summation or Davenport's lemma to [illegible] justify $\sum g\gamma_T(f) = \frac{1}{[\text{illegible}]}\int g\gamma_T(f)\,df + O(\dots)$

$$= \int \gamma_T(f)\,df \qquad \text{(III)}$$

The claim follows by ~~integrating~~ plugging (II)/(III) into (I).

Step 4 Show $\displaystyle\sum_{\substack{f\in V^{a\neq 0}(\mathbb{Z})\\ f\notin V^{i}(\mathbb{Z})}} \alpha_T(f) = o(T)$

If $f \bmod p \in V(\mathbb{F}_p)$ is irreducible for some p, then f is irreducible over $\mathbb{Q}$.

The claim follows from Step 3 using a sieve as in the proof of Thm 3.2.1.

Step 5 a) Show that $V^m(\mathbb{Z}_p) \subset V(\mathbb{Z}_p)$ is given by fin. many cong. cond. and b) compute $w_p := \mathrm{vol}(V^m(\mathbb{Z}_p))$.

$$GL_2(\mathbb{Z}_p)\backslash V^m(\mathbb{Z}_p) \longleftrightarrow \{\text{(rings of int. of) étale cubic ext. } L \text{ of } \mathbb{Q}_p\}/\cong$$

$$\mathrm{Stab}_{GL_2(\mathbb{Z}_p)}(f) \cong \mathrm{Aut}(L)$$

$$|\mathrm{disc}(f)|_p = |\mathrm{disc}(L)|_p$$

For each L, pick a corresponding $f_L \in V^m(\mathbb{Z}_p)$.

$$\Rightarrow V^m(\mathbb{Z}_p) = \bigsqcup_L GL_2(\mathbb{Z}_p) f_L \qquad (\mathrm{I})$$

Let $\eta_L : GL_2(\mathbb{Z}_p) \to V^m(\mathbb{Z}_p)$, $g \mapsto g f_L$. Each element of $GL_2(\mathbb{Z}_p) f$ has exactly $\#\mathrm{Aut}(L)$ preimages.

By Lemma 15.1.2, $|\mathrm{Jac}(\eta_L)(g)|_p = |\mathrm{disc}(f_L)|_p = |\mathrm{disc}(L)|_p$.

⟹ By change of variables (Thm 13.3),

$$\mathrm{vol}(GL_2(\mathbb{Z}_p)f_L) = \frac{|\mathrm{disc}(L)|_p}{\#\mathrm{Aut}(L)} \cdot \mathrm{vol}(GL_2(\mathbb{Z}_p)).$$

$$\underset{(I)}{\Longrightarrow} w_p = \mathrm{vol}(V^m(\mathbb{Z}_p)) = \sum_L \frac{|\mathrm{disc}(L)|_p}{\#\mathrm{Aut}(L)} \cdot \mathrm{vol}(GL_2(\mathbb{Z}_p))$$

$$\sum_L \frac{|\mathrm{disc}|}{\#\mathrm{Aut}} = 1 + \frac{1}{p} + \frac{1}{p^2}$$

↑ Mass formula (Thm 14.6)

$$\mathrm{vol}(GL_2(\mathbb{Z}_p)) = \left(1 - \frac{1}{p}\right) \cdot \mathrm{vol}(SL_2(\mathbb{Z}_p))$$

↑ Thm 7.5.2, Thm 7.5.3

$$\Rightarrow w_p = \left(1+\frac{1}{p}+\frac{1}{p^2}\right)\left(1-\frac{1}{p}\right) \cdot \mathrm{vol}(SL_2(\mathbb{Z}_p))$$

$$= \left(1 - \frac{1}{p^3}\right) \cdot \mathrm{vol}(SL_2(\mathbb{Z}_p)).$$

$$\underset{\text{Cor 15.5.8}}{\Longrightarrow} \prod_p w_p \cdot V = \prod_p \left(1 - \frac{1}{p^3}\right) \mathrm{vol}(SL_2(\mathbb{Z}_p)) \cdot \frac{1}{3} \mathrm{vol}(SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R}))$$

$$= \frac{1}{3} \prod_p \left(1 - \frac{1}{p^3}\right) = \frac{1}{3\zeta(3)}$$ as claimed

↑ Cor 7.5.4

For a), note that the image of the compact set $GL_2(\mathbb{Z}_p)$ ($GL_2(\mathbb{Z}_p)f_L$) is compact. $\Longrightarrow$ $V^m(\mathbb{Z}_p)$ is compact.

↑ $\#\{L\} < \infty$

Moreover, the image of $GL_2(\mathbb{Z}_p)$ is open by Hensel's lemma because the Jacobian det. $|disc(L)|_p$ of η_L is $\neq 0$.

$\Rightarrow V^{m}(\mathbb{Z}_p)$ is a compact and open subset of $V(\mathbb{Z}_p)$.

$\Rightarrow V^{m}(\mathbb{Z}_p)$ is defined by finitely many congruence conditions.