Let $GL_2(K)$ acts on $\mathbb{P}^1(K)$ by $M[x:y] = [x':y']$ where $\binom{x'}{y'} = (M^T)^{-1}\binom{x}{y}$.

This actually factors through an action of $PGL_2(K) = GL_2(K)/K^\times$.

<u>Lemma 15.3.1</u>

$PGL_2(K)$ acts simply and transitively on the set of (ordered!) triples (P_1, P_2, P_3) of distinct points in $\mathbb{P}^1(K)$.

<u>Pf</u> Let $P_i = [x_i : y_i]$, $\quad v_i := (x_i, y_i) \in K^2$

$$\left.\begin{array}{l} M[1:0] = P_1 \\ M[0:1] = P_2 \end{array}\right\} \Leftrightarrow (M^T)^{-1} = \begin{bmatrix} \lambda x_1 & \mu x_2 \\ \lambda y_1 & \mu y_2 \end{bmatrix} \text{ for some } \lambda, \mu \in K^\times.$$

Then, $M[1:1] = P_3 \iff \lambda v_1 + \mu v_2 = \tau v_3$ for some $\tau \in K^\times$.

Since any two of the vectors v_1, v_2, v_3 are linearly independent, there is a unique such triple (λ, μ, τ) up to mult. by $K^\times$.

$\square$

~~Cor 15.3.2 $\exists f [f]$ corr. to the set $A \subset \mathbb{P}^1(\bar{K})$ of roots, then~~

$\cancel{\mathrm{Stab}_{PGL_2(\bar{K})}(\underbrace{[f]}_{\bar{K}^\times \backslash \{f \in \mathbb{Z}[\bar{K}] \mid \operatorname{disc}(f) \neq 0\}}) = \mathrm{Stab}}$

Cor 15.3.2

$PGL_2(\overline{K})$ acts transitively on $\{A \subseteq \mathbb{P}^1(K) \mid \# A = 3\}$ with $\mathrm{Stab}_{PGL_2(K)}(A) \cong S_3$.

Pf The three points in A can be permuted. $\square$

Cor 15.3.3

$PGL_2(\overline{K})$ acts transitively on $K^\times \backslash \{f \in V(\overline{K}) \mid \mathrm{disc}(f) \neq 0\}$ with $\mathrm{Stab}_{PGL_2(\overline{K})}([f]) \cong S_3$.

Same for K^{sep} instead of $\overline{K}$.

Cor 15.3.4

$GL_2(\overline{K})$ acts transitively on $\{f \in V(\overline{K}) \mid \mathrm{disc}(f) \neq 0\}$ with $\mathrm{Stab}_{GL_2(\overline{K})}(f) \cong S_3$.

Pf This follows from the prev. cor together with

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} f = \lambda \cdot f.$$

$\square$

Rmk 15.3.5

If the three distinct roots of $f \in V(K)$ lie in $\mathbb{P}^1(K)$, then $\mathrm{Stab}_{GL_2(K)}(f) = \mathrm{Stab}_{GL_2(\overline{K})}(f) \cong S_3$.

15.4. Nonabelian group cohomology

Def Let G be a finite group. A <u>G-group</u> is a group A (not necessarily abelian!) with a left action of G (such that $g(a_1 a_2) = (ga_1)(ga_2)$).

We get the subgroup

$$H^0(G,A) = A^G = \{a \in A \mid ga = a \ \forall g \in G\}.$$

Let $Z^1(G,A)$ be the set of 1-cocycles: ~~maps~~ $\varphi : G \to A$ (not necessarily group hom.)

such that $\varphi(gh) = \varphi(g) \cdot g\varphi(h) \quad \forall g, h \in G$

Define an action of A on $Z^1(G,A)$ by

$$(a\varphi)(g) = a \cdot \varphi(g) \cdot (ga^{-1}) \quad \text{for } a \in A,\ \varphi \in Z^1(G,A),\ g \in G.$$

(check that $a\varphi \in Z^1(G,A)$!)

~~Let $B^1(G,A)$ be the~~

The <u>1-st cohomology set</u> is the set of orbits:

$$H^1(G,A) = A \backslash Z^1(G,A).$$

There is a special A-orbit $B^1(G,A) \subseteq Z^1(G,A)$, consisting of the <u>1-coboundaries</u>: maps of the form $(g \mapsto a \cdot (g^{-1}a))$ for some $a \in A$.

$\rightsquigarrow$ $H^1(G,A)$ is a pointed set with base point $B^1(G,A)$.

Rmk ~~...~~ Defining $H^2, H^3, \ldots$ is problematic!

~~...~~

Lemma 15.4.1

If G acts trivially on A ~~...~~, then

$$H^0(G,A)=A, \qquad Z^1(G,A)=\mathrm{Hom}_{\text{group}}(G,A),$$

and A acts on $Z^1(G,A)$ by conjugation: $(a\varphi)(g)=a\varphi(g)a^{-1}$.

~~...~~ Hence, $H^1(G,A)= A \underset{\substack{\uparrow \\ \text{conj.}}}{\backslash} \mathrm{Hom}_{\text{group}}(G,A)$

Rmk The pointed sets $H^1(G,A)$ satisfy functoriality, you get a truncated long exact sequence, ...

Reference 1) Milne: Algebraic groups, Lie groups, and their arithmetic subgroups, chapter VI.

2) Gille-Szamuely: Central simple algebras and Galois cohomology, section 2.3 (Galois descent)

Nonabelian Galois cohomology:

<u>Def</u> Let $L|K$ be a Galois ext. with Galois group G and let A be a G-group such that

~~every element of A is fixed~~

for every $a \in A$,

$$[G : \mathrm{Stab}_G(a)] < \infty .$$

("a is defined over a finite subext $F \subseteq L$ of K")

<u>Ex</u> $A = L$, $A = L^\times$, $A = GL_n(L)$, $A =$ any group with trivial G-action

<u>Def</u> (cont.)

$$H^0(L|K, A) := H^0(G, A) = A^G$$

If $L|K$ is a finite ext.,

$$H^1(L|K, A) := H^1(G, A).$$

For arbitrary $L|K$, let

$$H^1(L|K, A) := \varinjlim_{\substack{H \subseteq G \\ \text{normal} \\ \text{subgr.} \\ \text{with} \\ [G:H] < \infty}} H^1(G/H, A^H),$$

or define cocycles requiring that $\varphi : \underset{\text{Krull}}{G} \to \underset{\text{discrete top.}}{A}$ is continuous.

Phenomenon 15.4.2

~~Let E be something defined over K~~

~~...~~

For any object P defined over K, let P_L be the corresponding object over L. ("base change to L")

Then, we often have a bijection

$$H^1(L|K, \operatorname{Aut}(P_L)) \longleftrightarrow \{\text{objects } Q \text{ defined over } K \text{ with } P_L \underset{L}{\cong} Q_L\}/\cong$$

$$\left(\underset{\mathrm{Gal}(L|K)}{\sigma \in} \mapsto f\circ(\sigma \circ f^{-1})\right) \longleftarrow Q \text{ with } Q_L \xrightarrow[\sim]{f} P_L$$

$$B^1(L|K, \operatorname{Aut}(P_L)) \longmapsto P$$

Here, $\mathrm{Gal}(L|K)$ acts on isom. $Q_L \to P_L$ (and on automorphisms of P_L) by acting on the coefficients of the map. In other words, $\sigma f = \sigma\circ f\circ\sigma^{-1}$.

$$\varphi \longmapsto Q = \{x \in P_L \mid \sigma(x) = \varphi(\sigma)x \ \forall \sigma \in G\}$$

$$\boxed{\text{The crux is whether this actually gives back } Q \text{ with } P_L \cong Q_L.}$$

Moreover, if the 1-cocycle φ corr. to the object Q, then

$$\operatorname{Stab}_{\operatorname{Aut}(P_L)}(\varphi) \cong \operatorname{Aut}(P).$$

An example:

Thm 15.4.3 Consider n-dimensional vector spaces V over K.

~~Let $V = K^n$ (previously called D_2)~~

For any V, $V_L := V \otimes_K L$.

Take $V = K^n$.

$\rightsquigarrow$ $\mathrm{Aut}(V_L) = GL_n(L)$, with the obvious action of $G = \mathrm{Gal}(L|K)$.

We have a bijection

$$H^1(L|K, GL_n(L)) \longleftrightarrow \{n\text{-dim. vector spaces } W \text{ over } K \text{ with } V_L \cong W_L\}/\cong$$

$$(\sigma \mapsto M(\sigma M^{-1})) \longleftarrow W, \text{ with an isom. } W_L \xrightarrow{\sim} V_L$$

given by a matrix $M \in GL_n(L)$ w.r.t. choices of K-basis of V and W.

$\mathrm{Stab}_{GL_n(L)}(\varphi) \cong GL_n(K) \cong \mathrm{Aut}(K^n)$.

Cor 15.4.4 $H^1(L|K, GL_n(L)) = \{*\}$ since there is only one n-dim. vector space over K (up to $\cong$).

~~There is only one n-dim. vector space over K (up to $\cong$), so $H^1(L|K, GL_n(L)) = \{*\}$.~~

(For $n=1$, this fact $H^1(L|K, L^\times)$ is called Hilbert 90.)

Pf See the references 2 or 1. $\square$

Another example
Thm 13.4.5 Consider the étale degree n ext.$^{\underline{s}}$ V of K.

$$V_L := V \otimes_K L$$

Take $V = \underbrace{K \times \ldots \times K}_{n}$.

$\mathrm{Aut}(V_L) = S_n$ with the trivial action of G.

We have a bijection

$$\begin{array}{ccc} H^1(L|K, S_n) & \longleftrightarrow & \{\text{étale deg. } n \text{ ext. } W \text{ of } K \text{ with } V_L \cong W_L\}/\cong \\ \| & & \\ S_n \backslash \mathrm{Hom}(G, S_n) & & \end{array}$$

$\mathrm{Stab}_{S_n}(f) \cong \mathrm{Aut}(W)$.

Ex If $L = K^{sep}$, then RHS $= \{\text{étale deg. } n \text{ ext. of } K\}/\cong$.

(That's Thm 12.2!)