We discussed integration by substitution over nonarchimedean local fields last time. More generally, one can perform a change of variables in any dimension:

Theorem 13.3. *Let A be a compact open subset of K^n. Let $f_1, \dots, f_n \in K[X_1, \dots, X_n]$. For any $y \in K^n$, let $m(y) = \#\{x \in A \mid f(x) = y\}$. Then,*

$$\int_{K^n} m(y)\mathrm{d}y = \int_A |\det \mathrm{Jac}(f)(x)|\mathrm{d}x,$$

where $\mathrm{Jac}(f)(x) = (\frac{\partial f_i(x)}{\partial x_j})_{i,j}$ is the Jacobian matrix.

We skip the proof, which works similarly to Theorem 13.2, but using an n-dimensional form of Hensel's lemma.

Remark 13.4. *There is a good notion of manifolds over K. One can integrate real-valued functions over manifolds, and there is a corresponding change of variables formula. (See the two references mentioned last time: [Pop] and [Igu00].)*

14 Some mass formulas

One can either count isomorphism classes of (separable) field extensions of K, or subfields of K^{sep}. Of course, Galois conjugate subfields are isomorphic, so there may be fewer isomorphism classes than subfields of K^{sep}. More precisely:

Lemma 14.1. *Let L be a separable field extension of K of degree n. Then,*

$$\#\{K \subseteq L' \subseteq K^{\mathrm{sep}} \mid L' \cong L \text{ as } K\text{-algebras}\} = \frac{n}{\#\,\mathrm{Aut}(L)}.$$

Proof. There are n embeddings $L \hookrightarrow K^{\mathrm{sep}}$. Two embeddings ρ_1, ρ_2 have the same image if and only if $\rho_1 = \rho_2 \circ \sigma$ for some automorphism σ of L. □

For the rest of this section, let K be a nonarchimedean local field with residue field $\mathbb{F}_q$.

Theorem 14.2 (Serre's mass formula, [Ser78])**.** *Consider the totally ramified separable degree n field extensions L of K, up to isomorphism. We have*

$$\sum_L \frac{|\operatorname{disc}(L|K)|_K}{\#\,\mathrm{Aut}(L)} = \frac{1}{q^{n-1}}.$$

Remark 14.3. *Any inseparable extension L of K has $\operatorname{disc}(L|K) = 0$, so including them wouldn't change the sum.*

Remark 14.4. *There are infinitely many (separable) totally ramified degree n field extensions L of K if and only if the characteristic of K divides n.*

Proof. By Lemma 14.1, we can write the left-hand side as the following sum over totally ramified degree n field extensions $L \subseteq K^{\mathrm{sep}}$ of K:

$$\frac{1}{n} \cdot \sum_L |\operatorname{disc}(L|K)|.$$

For any L as above, let $U_L \subseteq L$ be the set of uniformizers in L. Let P be the set of separable monic degree n Eisenstein polynomials $f \in \mathcal{O}_K[X]$. The characteristic polynomial of any $a \in U_L$ lies in P since L is totally ramified. Conversely, the n roots of any $f \in P$ in K^{sep} each generate a totally ramified degree n extension of K. We thus have an n-to-1 map

$$\psi : \coprod_{\substack{L \subseteq K^{\mathrm{sep}} \\ \text{totally ramified} \\ \text{degree } n}} U_L \to P$$

sending $a \in U_L$ to its characteristic polynomial. We again identify monic degree n polynomials with their coefficient tuple, so $P \subseteq \mathcal{O}_K^n$.

The theorem will follow from the change of variables formula applied to this map.

We first compute the volume of P directly. The set of Eisenstein polynomials $X^n + c_{n-1}X^{n-1} + \cdots + c_0$ (with $c_0 \in \pi_K \mathcal{O}_K^\times$ and $c_1, \ldots, c_{n-1} \in \pi_K \mathcal{O}_K$) has volume $q^{-n}(1-q^{-1})$. The set of inseparable monic degree n polynomials f in $\mathcal{O}_K[X]$ has volume 0 because all inseparable polynomials f have discriminant zero. (The discriminant is a nonzero polynomial in the coefficients of f. The set of roots of any nonzero polynomial has volume 0.) Hence,

$$\operatorname{vol}(P) = q^{-n}(1 - q^{-1}).$$

Fix a field L as above, and any uniformizer π_L of L. (As L is totally ramified, we have $v_K(\pi_L) = \frac{1}{n} v_K(\pi_K)$.) Our goal is to compute the volume of the image of U_L. Note that $(1, \pi_L, \ldots, \pi_L^{n-1})$ is an integral basis of L. The map $d : K^n \to L$, $(b_0, \ldots, b_{n-1}) \mapsto b_0 + b_1\pi_L + \cdots + b_{n-1}\pi_L^{n-1}$ therefore sends $\mathcal{O}_K^n$ to $\mathcal{O}_L$. Our Haar measure on K^n corresponds to our Haar measure on L under this map. The uniformizers of L are exactly the linear combinations $b_0 + b_1\pi_L + \cdots + b_{n-1}\pi_L^{n-1}$ with $b_0 \in \pi_K \mathcal{O}_K$ and $b_1 \in \mathcal{O}_K^\times$ and $b_2, \ldots, b_{n-1} \in \mathcal{O}_K$. Hence,

$$\operatorname{vol}(U_L) = q^{-1}(1 - q^{-1}).$$

Consider the n homomorphisms $\rho_1, \ldots, \rho_n : L \to K^{\mathrm{sep}}$, and combine them to a map $\rho : L \to (K^{\mathrm{sep}})^n$. The linear map $\rho \circ d : K^n \to (K^{\mathrm{sep}})^n$ is described by the matrix $(\rho_i(\pi_L^j))_{i,j}$. Since $(1, \pi_L, \ldots, \pi_L^{n-1})$ is an integral basis of L, its determinant is $|\operatorname{disc}(L|K)|^{1/2}$.

As in section 9, we consider the map

$$\chi : (K^{\mathrm{sep}})^n \to \{f \in K^{\mathrm{sep}}[X] \text{ monic, degree } n\} \cong K^n$$

that sends $a = (a_1, \ldots, a_n)$ to $(X - a_1) \cdots (X - a_n)$. Its Jacobian determinant has norm $\prod_{i<j} |a_i - a_j|$. (See Lemma 9.4.) If $a = \rho(\pi'_L)$ for a uniformizer π'_L of L, then this product is $|\operatorname{disc}(\pi'_L)|^{1/2} = |\operatorname{disc}(L|K)|^{1/2}$, again because $(1, \pi'_L, \ldots, \pi'^{n-1}_L)$ is an integral basis.

The composition $\chi \circ \rho \circ d : K^n \to (K^{\mathrm{sep}})^n$ sends $(b_0, \ldots, b_{n-1})$ to (the coefficient tuple of) the characteristic polynomial of $b_0 + b_1\pi_L + \cdots + b_{n-1}\pi_L^{n-1}$. Combining the above computations, we see that the norm of the Jacobian determinant of this map is $|\operatorname{disc}(L|K)|$.

Hence, by Theorem 13.3, if we interpret the image $\psi(U_L)$ as a multiset, then

$$\operatorname{vol}(\psi(U_L)) = |\operatorname{disc}(L|K)| \cdot \operatorname{vol}(U_L) = |\operatorname{disc}(L|K)| \cdot q^{-1}(1 - q^{-1}).$$

As ψ is n-to-1, we have

$$\sum_{\substack{L \subseteq K^{\mathrm{sep}} \\ \text{totally ramified} \\ \text{degree } n}} \operatorname{vol}(\psi(U_L)) = n \cdot \operatorname{vol}(P),$$

so

$$\sum_L |\operatorname{disc}(L|K)| \cdot q^{-1}(1-q^{-1}) = n \cdot q^{-n}(1-q^{-1}),$$

so indeed

$$\frac{1}{n} \cdot \sum_L |\operatorname{disc}(L|K)| = q^{-(n-1)}. \qquad \square$$

Corollary 14.5. *Consider the separable field extensions L of K with ramification index e and inertia degree f, up to isomorphism. We have*

$$\sum_L \frac{|\operatorname{disc}(L|K)|}{\#\operatorname{Aut}(L)} = \frac{1}{f \cdot q^{(e-1)f}}.$$

Proof. To avoid confusion, we will write $|\cdot|_K$ and $|\cdot|_L$ for the normalized norm on K and L, respectively, and similarly q_K and q_L for the residue field size of K and L, respectively.

Using, Lemma 14.1, the left-hand side can again be rewritten as a sum over field extensions $L \subseteq K^{\mathrm{sep}}$ of K with ramification index e and inertia degree f:

$$\frac{1}{ef} \cdot \sum_{L \subseteq K^{\mathrm{sep}}} \frac{|\operatorname{disc}(L|K)|_K}{\#\operatorname{Aut}(L)}.$$

Each such field extension $L|K$ decomposes uniquely as $L|F|K$ with $F|K$ unramified of degree f and $L|F$ totally ramified of degree e. (Here, F is the splitting field of the polynomial $X^{q^f} - X$.) By the relative discriminant formula,

$$|\operatorname{disc}(L|K)|_K = |\operatorname{Nm}_{F|K}(\operatorname{disc}(L|F)) \cdot \operatorname{disc}(F|K)|_K = |\operatorname{Nm}_{F|K}(\operatorname{disc}(L|F))|_K = |\operatorname{disc}(L|F)|_L.$$

Since there is exactly one unramified extension $F \subseteq K^{\mathrm{sep}}$ of degree f, the theorem implies:

$$\frac{1}{ef} \cdot \sum_{L \subseteq K^{\mathrm{sep}}} |\operatorname{disc}(L|K)|_K = \frac{1}{f} \cdot \frac{1}{q_L^{e-1}} = \frac{1}{f \cdot q_K^{(e-1)f}}. \qquad \square$$

We can now prove the following mass formula regarding all étale extensions of K.

Theorem 14.6 ([Bha07, Theorem 1.1] and [Ked07, Theorem 1.1])**.** *Consider the étale K-algebras L of degree n, up to isomorphism. We have*

$$\sum_L \frac{|\operatorname{disc}(L|K)|}{\#\operatorname{Aut}(L)} = \sum_{r=0}^{n} \frac{P(n,r)}{q^{n-r}},$$

where $P(n,r)$ is the number of partitions of the integer n into r positive summands.

Example 14.7. *If $2 \nmid q$, then the degree 2 extensions are $K \times K$, $K(\sqrt{a})$, $K(\sqrt{\pi_K})$, $K(\sqrt{a\pi})$, where $a \in \mathcal{O}_K^\times$ is a quadratic nonresidue. They all have two automorphisms, and their discriminant norms are $1, 1, q^{-1}, q^{-1}$, respectively. Hence, $\sum_L \frac{|\operatorname{disc}(L|K)|}{\#\operatorname{Aut}(L)} = 1 + q^{-1}$.*

Proof. Any L can be written as $L = L_1 \times \cdots \times L_r$, with $\operatorname{disc}(L|K) = \operatorname{disc}(L_1|K) \cdots \operatorname{disc}(L_r|K)$, $n = [L_1 : K] + \cdots + [L_r : K]$. Consider the obvious action of S_r on the set of tuples $(L_1, \ldots, L_r)$ of isomorphism classes of field extensions of K. We have

$$\#\operatorname{Aut}(L) = \#\operatorname{Aut}(L_1) \cdots \#\operatorname{Aut}(L_r) \cdot \#\operatorname{Stab}_{S_r}((L_1, \ldots, L_r)).$$

(Any automorphism consists of a permutation of isomorphic factors of L together with isomorphisms of the individual factors.)

Let

$$a_n := \sum_{\substack{L \\ \text{separable field ext.} \\ \text{of degree } n}} \frac{|\operatorname{disc}(L|K)|}{\#\operatorname{Aut}(L)}.$$

It follows from the above discussion that

$$b_n := \sum_{\substack{L \\ \text{étale } K\text{-algebra} \\ \text{of degree } n}} \frac{|\operatorname{disc}(L|K)|}{\#\operatorname{Aut}(L)} = \sum_{r\geqslant 0} \sum_{\substack{S_r\text{-orbit of } (L_1,\dots,L_r) \\ \text{with } n=\sum_i [L_i:K]}} \prod_i \frac{|\operatorname{disc}(L_i|K)|}{\#\operatorname{Aut}(L_i)} \cdot \frac{1}{\#\operatorname{Stab}_{S_r}((L_1,\dots,L_r))}.$$

By the orbit-stabilizer theorem, this is

$$\sum_{r\geqslant 0} \frac{1}{r!} \sum_{\substack{(L_1,\dots,L_r) \\ \text{with } n=\sum_i [L_i:K]}} \prod_i \frac{|\operatorname{disc}(L_i|K)|}{\#\operatorname{Aut}(L_i)}.$$

This implies that the generating functions $\sum_n a_n X^n$ and $\sum_n b_n X^n$ are related by the power series identity

$$\sum_{n\geqslant 0} b_n X^n = \exp\left(\sum_{n\geqslant 0} a_n X^n\right).$$

According to the previous corollary, we have

$$a_n = \sum_{\substack{e,f\geqslant 1: \\ ef=1}} \frac{1}{f\cdot q^{(e-1)f}},$$

so

$$\sum_{n\geqslant 0} a_n X^n = \sum_{e,f\geqslant 1} \frac{X^{ef}}{f\cdot q^{(e-1)f}} = -\sum_{e\geqslant 1} \log\left(1-\frac{X^e}{q^{e-1}}\right).$$

Hence,

$$\sum_{n\geqslant 0} b_n X^n = \prod_{e\geqslant 1} \frac{1}{1-\frac{X^e}{q^{e-1}}} = \prod_{e\geqslant 1}\sum_{t\geqslant 0}\left(\frac{X^e}{q^{e-1}}\right)^t = \sum_{t_1,t_2,\dots\geqslant 0} \frac{X^{\sum_{e\geqslant 1} et_e}}{q^{\sum_{e\geqslant 1}(e-1)t_e}} = \sum_{n\geqslant 0} \frac{P(n,r)}{q^{n-r}} X^n.$$

(Any choice of $t_1, t_2, \dots$ with $n = \sum_{e\geqslant 1} et_e$ corresponds to a partition of n into $\sum_{e\geqslant 1} t_e$ summands, where e occurs t_e times.) □

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