Each base point $x_0 \in X$ gives you a function $\beta = \beta_{x_0}$

such that $\displaystyle\sum_{z \in Hy} \beta_{x_0}(z) = \mathrm{vol}(\mathrm{Stab}_H(y) \backslash \mathrm{Stab}_G(y))$ $\forall y \in X$.

Idea: To smoothen β, average over $x_0 \in X$ using a smooth weight $\eta(x_0)$:

$$\tilde{\beta}(z) := \int_X \beta_{x_0}(z)\, \eta(x_0)\, dx_0$$

$$= \int_X \int_{\substack{\mathrm{Stab}_G(x_0) \\ \text{with } z = g x_0}} \alpha(gs)\, d_{x_0}s\, \eta(x_0)\, dx_0$$

$$\underset{g' = g^{-1}}{=} \int_G \alpha(g'^{-1})\, \eta(g'^{-1} z)\, dg'$$

$= \text{average over } \underset{\text{in fund. domain}}{g'} \text{ of } \underbrace{g' \text{ translates of } \eta}_{\text{count points in here!}}.$

9. Counting algebraic integers

~~Let $\overline{\mathbb{Z}} \subset \overline{\mathbb{Q}}$ be the set of alg. integers.~~

Def $\deg(\alpha) = \deg(\text{min. pol. of } \alpha)$ for $\alpha \in \overline{\mathbb{Q}}$.

Def The *length* of $\alpha \in \overline{\mathbb{Q}}$ is

$$|\alpha| = \max_{\sigma : \overline{\mathbb{Q}} \to \mathbb{C}} \underbrace{|\sigma(\alpha)|}_{\substack{\text{the usual} \\ \text{absolute value (magnitude)} \\ \text{on } \mathbb{C}}} .$$

Rmk This agrees with the def. of $|\alpha|$ for $\alpha \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ in section 4.3.

Def Let $\overline{\mathbb{Z}}$ be the set of alg. integers in $\overline{\mathbb{Q}}$.
Let $\overline{\mathbb{Z}}_n = \{\alpha \in \overline{\mathbb{Z}} \text{ of degree } n\}$.

Goal Count $\alpha \in \overline{\mathbb{Z}}_n$ with $|\alpha| \leq T$.

Idea Count minimal polynomials.

Def Let $\chi : \mathbb{C}^n \longrightarrow \left\{ \begin{array}{c} f \in \mathbb{C}[X] \\ \text{monic deg. } n \end{array} \right\} \cong \mathbb{C}^n$.

$$(a_1, \dots, a_n) \mapsto \prod_{i=1}^{n} (X - a_i)$$

$$X^n + c_{n-1}X^{n-1} + \dots + c_0 \mapsto (c_0, \dots, c_{n-1})$$

Thm 9.1 For any $n \geq 1$, there is a constant $C_n > 0$ such that $\#\{\alpha \in \overline{\mathbb{Z}}_n \mid |\alpha| \leq T\} \underset{n}{\sim} C_n T^{n(n+1)/2}$.

Ex $n = 1 \Rightarrow \overline{\mathbb{Z}}_n = \mathbb{Z}$

LHS $\sim 2T$.

More precisely:

Thm 9.2 For any $r_1, r_2 \geq 0$ with $r_1 + 2r_2 = n$, there is a constant $C_{r_1,r_2} > 0$ s.t. $\#\{\alpha \in \overline{\mathbb{Z}}_n \text{ of signature } (r_1,r_2) \mid |\alpha| \leq T\} \sim C_{r_1,r_2} T^{n(n+1)/2}$

Pf

$\boxed{\text{let } A = \{a \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \mid |a| \leq 1, \ \rho_i(a) \neq \rho_j(a) \ \forall i \neq j\}, \quad A_T = T \cdot A = \{a \mid |a| \leq T, \ \rho_i(a) \neq \rho_j(a) \ \forall i \neq j\}.}$

let $\rho_1, \dots, \rho_n$ be $\underbrace{\text{the hom.}}_{n\ \mathbb{R}\text{-algebra}}$ $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \to \mathbb{C}$ and let

$$\rho : \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \to \mathbb{C}^n, \quad a \mapsto (\rho_1(a), \dots, \rho_n(a)).$$

Let $\psi : \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \longrightarrow \{f \in \mathbb{R}[x] \text{ monic, deg. } n\} \cong \mathbb{R}^n$,

$$\psi(a) = \chi(\rho(a)).$$

We obtain an n-to-1 map

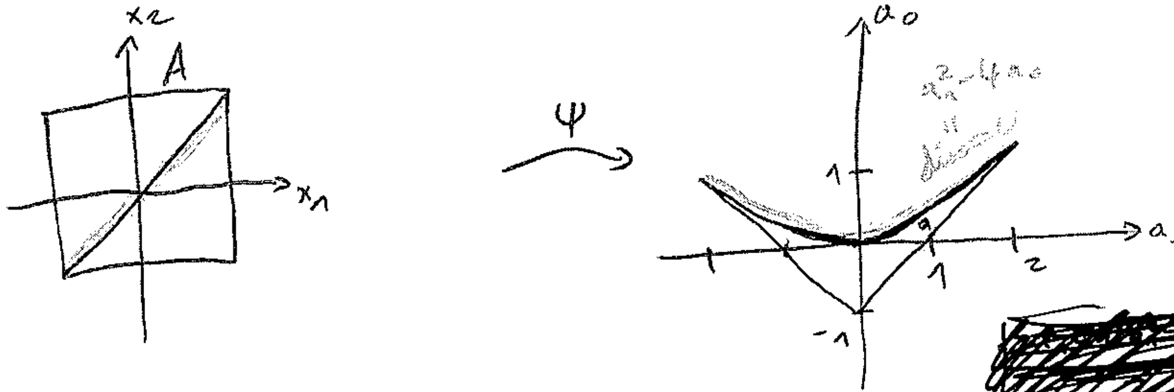
$$\{\alpha \in \overline{\mathbb{Z}} \text{ of signature } (r_1,r_2) \text{ with } |\alpha| \leq T\} \longrightarrow \{f \in \mathbb{Z}[x] \cap \psi(A_T) \text{ irreducible}\}$$

Note: If $\psi(a) = (c_{n-1}, \dots, c_0)$, then $\psi(Ta) = (Tc_{n-1}, \dots, T^n c_0)$.

$$\Rightarrow \psi(A_T) = \underbrace{\begin{pmatrix} T & & \\ & \ddots & \\ & & T^n \end{pmatrix}}_{D_T} \psi(A).$$

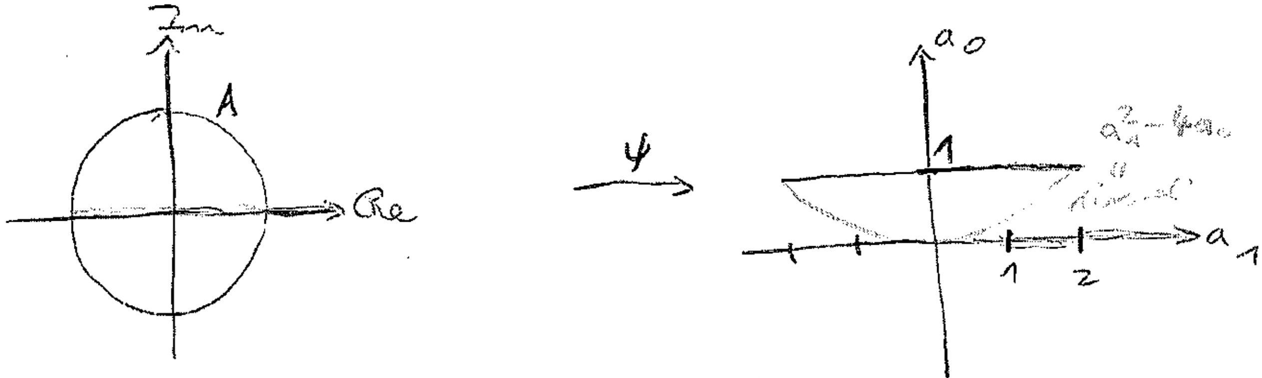
Ex Signature (2,0):

The Jacobian of $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $(x_1,x_2) \longmapsto (-(x_1+x_2),\, x_1x_2)$ has absolute determinant $|x_1-x_2|$. We have $\mathrm{vol}(\psi(A)) = \frac{4}{3}$, so $C_{2,0} = \frac{8}{3}$.



Ex Signature (0,1):

The Jacobian of $\psi : \mathbb{C} \longrightarrow \mathbb{R}^2$, $a+bi \longmapsto (-2a,\, a^2+b^2)$ has abs. determinant $4|b|$. We have $\mathrm{vol}(\psi(A)) = \frac{8}{3}$, so $C_{0,1} = \frac{16}{3}$.



Ex $C_2 = 8$

$\Rightarrow \#\{\alpha \in \overline{\mathbb{Z}}$ of signature (r_1, r_2) with $|\alpha| \leq T\}$

$$= \frac{1}{n} \cdot \#\{ f \in \mathbb{Z}[x] \cap D_T \psi(A) \text{ irreducible} \}$$

We have $\#\{f \in \mathbb{Z}[x] \cap D_T \psi(A)\} \sim T^{1+\dots+n} \cdot \mathrm{vol}(\psi(A))$. (I)

By the sieve theory argument in Thm 3.2.1,

$$\#\{ \underset{x^n + c_{n-1}x^{n-1} + \dots + c_0}{\overset{\shortparallel}{f}} \in \mathbb{Z}[x] \mid c_{n-i} \ll T^i \ \forall i \} = o(T^{1+\dots+n}).$$

$\square$

To prove (I), you can use for example Davenport's Lemma or:

Lemma 9.3 Let $A \subseteq \mathbb{R}^n$ be bounded with $\mathrm{vol}(\mathrm{int}(A)) = \mathrm{vol}(\overline{A}) = V$.

Let Λ be a full lattice in $\mathbb{R}^n$ with (Euclidean) succ. min. $\lambda_1, \dots, \lambda_n$. Then,

$$\#(\Lambda \cap A) \underset{n,A}{\sim} \frac{V}{\mathrm{covol}(\Lambda)} \quad \text{as } \lambda_n \to 0.$$

Pf ~~For the lower bound, approximate~~
Approximate 1_A from below and above by smooth compactly supported functions and apply Poisson summation (Thm 4.2.6).

To do this, fix a sm. fct. $\eta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ with $\mathrm{supp}(\eta) \subseteq D(1)$ and $\int \eta(x)\,dx = 1$.

Let $\eta_\delta(x) = \delta^n \eta(x/\delta)$. $\Rightarrow \mathrm{supp}(\eta_\delta) \subseteq D(\delta)$, $\int \eta_\delta(x)\,dx = 1$.

For any $\delta > 0$, let

$$U_\delta = \{ x \in \mathbb{R}^n \mid x + D(\varepsilon) \subseteq \mathrm{int}(A) \}$$

$$K_\delta = \overline{A} + D(\varepsilon).$$

$$\Rightarrow 0 \leq 1_{U_\delta} * \eta_\delta \leq 1_A \leq 1_{K_\delta} * \eta_\delta,$$

$1_{U_\delta} \xrightarrow{\delta \to 0} 1_{\mathrm{int}(A)}$ ptwise, increasing

$1_{K_\delta} \xrightarrow{\delta \to 0} 1_{\overline{A}}$ ptwise, decreasing

By Thm 4.2.6,

$$\sum_{x \in \Lambda} (1_{U_\delta} * \eta_\delta)(x) \sim \frac{\int 1_{U_\delta} * \eta_\delta}{\mathrm{covol}(\Lambda)} = \frac{\mathrm{vol}(U_\delta)}{\mathrm{covol}(\Lambda)} \xrightarrow{\delta \to 0} \frac{\mathrm{vol}(\mathrm{i}}{\mathrm{covol}}$$

by monotone convergence

$$\sum (1_{K_\delta} * \eta_\delta)(x) \sim \dots \frac{\mathrm{vol}(K_\delta)}{\mathrm{covol}(\Lambda)} \longrightarrow \frac{\mathrm{vol}(\overline{A})}{\mathrm{covol}(\Lambda)}.$$

You can show that $\varphi(A)$ satisfies the conditions of Lemma 9.3 and compare its volume using:

<u>Lemma 9.4</u> The Jacobian determinant of $\mathcal{V}: \mathbb{C}^n \to \mathbb{C}^n$ is $\pm \prod_{i<j} (a_i - a_j)$.

<u>Pf</u> Let $\partial_i = \frac{\partial}{\partial a_i}$.

The i-th row of the Jacobian ($J_n(a_1, \dots, a_n)$) is the coefficient vector of the polynomial

$$\frac{\partial \mathcal{V}}{\partial a_i}(a) = -\prod_{j \neq i} (X - a_j) \text{ of degree } n.$$

Subtract the n-th row from all other rows.

$$\frac{\partial \mathcal{V}}{\partial a_i}(a) - \frac{\partial \mathcal{V}}{\partial a_n}(a) = -(a_n - a_i) \cdot \underbrace{\prod_{j \neq i,n} (X - a_j)}_{\text{pol. of degree } n-2}.$$

Then, divide the i-th row by $a_n - a_i$.
The X^{n-1}-coeff. of $\frac{\partial \mathcal{V}}{\partial a_n}(a)$ is -1.

$\Rightarrow$ The resulting matrix looks like

$$\begin{bmatrix} 0 & \\ \vdots & \boxed{J_{n-1}(a_1, \dots, a_{n-1})} \\ 0 & \\ -1 & * \; * \quad * \end{bmatrix}.$$

$\Rightarrow \det(J_n(a_1, \dots, a_n)) = \pm \prod_{i<n} (a_i - a_n) \cdot \det(J_{n-1}(a_1, \dots, a_{n-1}))$.

$\square$

Lemma 9.5 The Jacobian determinant of $\rho : \underset{\substack{\cong \\ \mathbb{R}^n}}{\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}} \to \mathbb{C}^n$

is $\pm 2^{r_2}$.

Pf $\begin{array}{c}\mathbb{R} \longrightarrow \mathbb{R} \\ a \longmapsto a\end{array}$ has det 1.

$\begin{array}{c}\mathbb{C} \longrightarrow \mathbb{C}^2 \\ a \longmapsto (a, \bar{a})\end{array}$ has det 2.

$\square$

Cor 9.6 $\displaystyle vol(\psi(A)) = \frac{\cancel{2^{r_2}}}{r_1! \cdot \cancel{2^{r_2}} r_2!} \cdot \int_A \prod_{i<j} |\rho_i(a) - \rho_j(a)| \, da.$

Pf $A \longrightarrow \psi(A)$ is a $r_1! \cdot 2^{r_2} r_2!$-to-1 map, so the result follows from change of variables. $\square$

AS, 39

Counting only polynomials with $a_{n-1}=0$:

Thm 9.7 Fix some $n \geq 2$. There is a constant $C_n' > 0$ such that for all $t \in \mathbb{Z}$,

$$\#\{\alpha \in \overline{\mathbb{Z}} \text{ of degree } n \text{ and length } |\alpha| \leq T \text{ and trace } t\} \underset{t,n}{\sim} C_n' \cdot T^{(n-1)(n+2)/2}.$$

"Pf" $2 + \dots + n = \frac{(n-1)(n+2)}{2}$. $\square$