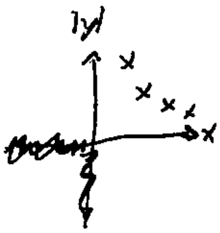
Pf (sketch) Let Λ be a full lattice in $\mathbb{R}^2$. Assume Λ contains no nonzero vectors of the form $(x,0)$ or $(0,y)$.

Let $B = \{(x,y)\in\Lambda \mid x>0,\ \nexists (x',y')\in\Lambda : 0<x'<x,\ |y'|\le|y|\}$

Clearly, $B \neq \emptyset$.

$v_1, v_2 \in B$, $v_1=(x_1,y_1)$, $v_2=(x_2,y_2)$

If , then either

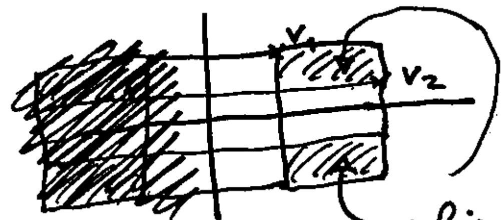


a) $\{x_1\} < \{x_2\}$ and $|y_1|>|y_2|$ (write this as $v_1 < v_2$)

or

b) $\{x_1\} > \{x_2\}$ and $|y_1|<|y_2|$ ($v_1 > v_2$)

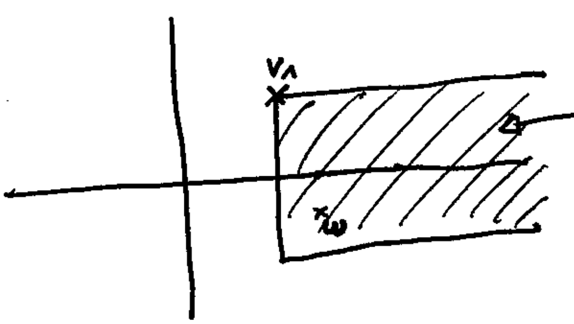
SKIP [For any $v_1, v_2 \in A$, there are only finitely many $w \in L$ with $v_1 < w < v_2$.



]

For every $v_1 \in A$, there exists a smallest $v_2 \in A$ with $v_2 > v_1$
and a largest $v_3 \in B$ with $v_3 < v_1$

~~(by Minkowski's first theorem)~~



~~Say v, w have positive x-coordinates.~~

~~Then, their y-coordinates have opposite signs:~~

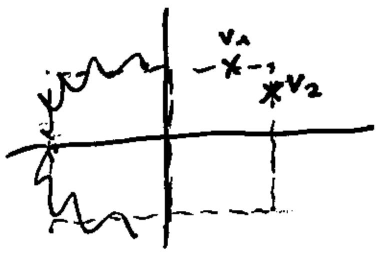
~~Otherwise, $w - v_1$ would have x-coord <~~

Claim Let $v_i = (x_i, y_i)$. Then, y_1, y_2 have opposite signs:

Pf Say $y_1, y_2 > 0$. Then, $v_2 - v_1 = (x_2 - x_1, y_2 - y_1) \in \Lambda$

has $0 < x_2 - x_1 < x_2$

and $|y_2 - y_1| \leq y_1$. ↯



Claim v_1, v_2 form a basis of Λ. other

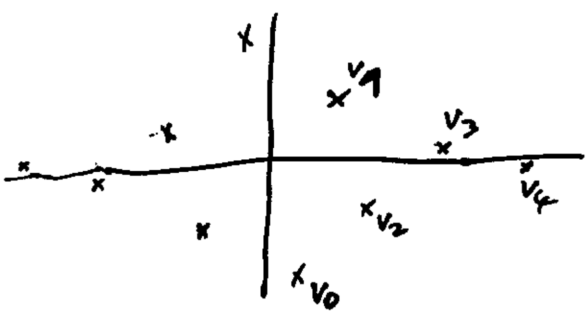
Pf The triangle $(0, v_1, v_2)$ contains no el. of Λ.

$\Rightarrow$ The parallelogram spanned by v_1, v_2 contains no lattice points besides $0, v_1, v_2, v_1 + v_2$. □

Conclusion:

B $= \{ \ldots, v_{-1}, v_0, v_1, \ldots \}$ with $\ldots < v_{-1} < v_0 < v_1 < v_2 < \ldots$

and, {assuming the x-coord. are > 0}, the y-coord. have alternating signs.



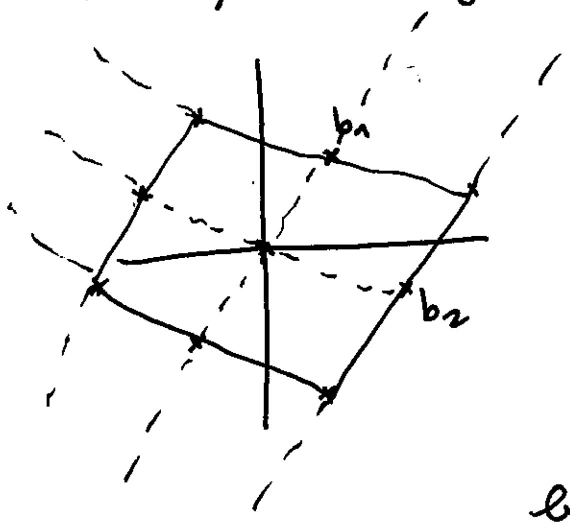
There is usually a unique index n such that

v_n lies in $\{(x,y) \mid |x| \leq |y|\}$ and v_{n+1} lies in $\{(x,y) \mid |x| \geq |y|\}$.

Then, $\begin{pmatrix} -v_n- \\ -v_{n+1}- \end{pmatrix}$ lies in A and in the $GL_2(\mathbb{Z})$-orbit corresponding to Λ.

Conversely, say (b_1, b_2) is a basis of Λ with $\begin{pmatrix} -b_1- \\ -b_2- \end{pmatrix} \in A_4$

~~The interior of the convex set spanned by $\pm b_1 \pm b_2$ contains no nonzero lattice points.~~



~~The parallelograms in this picture contain no lattice points in their interior.~~

Then, b_1, b_2 must be consecutive vectors in $\ldots, v_0, v_1, \ldots$ because there is no lin. combo $\underset{(0,0) \neq (x,y)}{=} c_1 b_1 + c_2 b_2 \in$ with $x < x_2$, $|y| < |y_1|$.

$\Downarrow$ c_1, c_2 have different signs

$\Downarrow$ c_1, c_2 have same sign

$\square$

Rmk You can compute v_{n+2} from v_n, v_{n+1} as follows:

$$v_{n+2} = v_n + k v_{n+1}, \text{ where } 1 \le k \in \mathbb{Z} \text{ is chosen so that}$$

y_{n+2} has the opposite sign as y_{n+1} and smaller magnitude:

~~If $y_n > 0$, $y_{n+1} < 0$, say, then $k = \left\lceil \frac{y_n}{-y_{n+1}} \right\rceil$~~

$$k = \left\lceil -\frac{y_n}{y_{n+1}} \right\rceil. \qquad \leadsto \frac{y_{n+2}}{y_{n+1}} = \frac{y_n}{y_{n+1}} + \left\lceil -\frac{y_n}{y_{n+1}} \right\rceil$$

~~(smells like continued fraction)~~

Rmk This gives you the "continued fraction" algorithm for computing a fund. unit of $\mathcal{O}_L$:

Start with a basis (v_0, v_1) of any fractional ideal $\mathfrak{a} \subseteq L \subseteq \mathbb{R}\times\mathbb{R}$.

Compute $v_2, v_3, \ldots$

If $u \in \mathcal{O}_L^\times$, then $uv_0 \in u\mathfrak{a} = \mathfrak{a}$ must also lie in B ~~(uv_0, uv_1) is also a reduced basis of $u\mathfrak{a} = \mathfrak{a}$.~~ $\Rightarrow uv_0 = v_p$ for some $p \in \mathbb{Z}$.

(and then $uv_i = v_{p+i}$ $\forall i \in \mathbb{Z}$)

$\Rightarrow$ The first unit in the sequence $\frac{v_1}{v_0}, \frac{v_2}{v_0}, \ldots$ is a fund. unit.

Lemma 8.13 Let α be the corresponding fund. dom. for $GL_2(\mathbb{Z}) \curvearrowright GL_2^{\pm 1}(\mathbb{R})$ and define β as in Lemma 8.11.

Let $f = aX^2 + bXY + cY^2 \in \mathcal{V}(\mathbb{R})$ with $\operatorname{disc}(f) = D$.

We have $\beta(f) \neq 0$ and only if $a \geq 0, c \leq 0, b \geq |a+c|$

Then, $\beta(f) = C \cdot \log \dfrac{\sqrt{D}+b}{\sqrt{D}-b}$. [This ... if $ac = 0$, but then D is a square!]

Pf Assume $a \neq 0$. Note that $f = g' \sqrt{D}XY$ for

$$g' = \begin{bmatrix} 1 & 1 \\ \frac{b \pm \sqrt{D}}{2a} & \frac{b \mp \sqrt{D}}{2a} \end{bmatrix} \in GL_2(\mathbb{R})$$

$$\Rightarrow f = g\sqrt{D}XY \text{ with } g = \frac{g'}{\sqrt{|\det(g')|}} \in GL_2^{\pm 1}(\mathbb{R}).$$

Recall: $\mathrm{Stab}_{GL_2^{\pm}(\mathbb{R})}(\sqrt{D}XY) = \{ \begin{pmatrix} s & \\ & t \end{pmatrix} \mid s, t \in \mathbb{R}^\times, st = \pm 1 \}$.

We have $g \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix} \in A$ if and only if

$s > 0$, $t \geq 0$, $t \geq s$, $(-b+\sqrt{D}) \cdot s \geq (b+\sqrt{D}) \cdot t$,

$2a \leq -b+\sqrt{D}$, $b+\sqrt{D} \leq 2a$

$s > 0$, $s \leq |t|$, $b \leq (\sqrt{D}-b) \cdot |t| \leq (\sqrt{D}+b) \cdot s$,

$2a \leq b+\sqrt{D}$, $\sqrt{D}-b \leq 2a$.

$a \geq 0$, $c \leq 0$, $b \geq |a+c|$, $1 \leq \frac{|t|}{s} \leq \frac{\sqrt{D}+b}{\sqrt{D}-b}$

$= \frac{1}{s^2}$

~~This is equiv. to~~

$$\cancel{|\sqrt{D}-2a| \le b \le \sqrt{D} \text{ and } s>0 \text{ and } 1 \le \frac{|t|}{s} \le \frac{\sqrt{D}+b}{\sqrt{D}-b}.}$$

$$\cancel{\frac{|t|}{s} = s^{-2} \text{ if } \begin{pmatrix} s & \\ & t \end{pmatrix} \in GL_2^{\pm 1}(\mathbb{R})}$$

$$\int_{\substack{GL_2^{\pm 1}(\mathbb{R}) \\ s>0, \\ 1 \le \frac{|t|}{s} \le \frac{\sqrt{D}+b}{\sqrt{D}-b}}} d^\times \begin{pmatrix} s & \\ & t \end{pmatrix} = C \cdot \log \frac{\sqrt{D}+b}{\sqrt{D}-b}.$$

$\square$

The cases $a=0$ works similarly... (?)

<u>Summary</u> If $D>0$ is the disc. of a q.r.l. L, then

$$h_L R_L = C' \sum_{\substack{f \in V(\mathbb{Z}) \\ \mathrm{disc}(f)=D}} \beta(f) = C'' \sum_{\substack{f \in V(\mathbb{Z}) \\ \mathrm{disc}(f)=D \\ |\sqrt{D}-2a| \le b \le \sqrt{D} \\ a \ge 0}} \log \frac{\sqrt{D}+b}{\sqrt{D}-b}.$$

This can be used to show Thm 8.9:

$$\sum_{0<\mathrm{disc}(L) \le T} h_L R_L \sim C''' T^{3/2}.$$

(For example, use a variant of Davenport's Lemma.)

Let $G, H, X, \alpha, \beta, \ldots$ as in Lemma 8.11. Assume X is a measure space with measure dx

Let dg be a measure on G such that

$$\int_G \rho(g)\,dg = \int_X \int_{\mathrm{Stab}_G(x_0)} \rho(gs)\,d_{x_0}s\,dx$$

for all reasonable g and all $x_0 \in X$.

where we write $x = gx_0$

$\rightsquigarrow$ In particular, $\int_X \beta(x)\,dx = \int_G \alpha(g)\,dg$.