<u>Reminder</u> $\underset{\text{disc}(f) = D = \text{disc}(L)}{L \text{ quadr. n. f.}}$ $\rightsquigarrow L = \mathbb{Q}[X]/(X^2-D)$

$$\cancel{\mathrm{Stab}_{GL_2(\mathbb{Z})}(f) \cong \mathcal{O}_L^\times}$$

$$L \otimes \mathbb{R} = \mathbb{R}[X]/(X^2-D) = \begin{cases} \mathbb{C}, & D<0 \\ \mathbb{R}\times\mathbb{R}, & D>0 \end{cases}$$

$$\cancel{\mathrm{Stab}_{GL_2(\mathbb{R})}(f)}$$

$$\begin{tikzcd}
\mathrm{Stab}_{GL_2(\mathbb{Z})}(f) \arrow[r, "\sim"] \arrow[d, hook] & \mathcal{O}_L^\times \arrow[d, hook] \\
\mathrm{Stab}_{GL_2^{\pm 1}(\mathbb{R})} \arrow[r, "\sim"] \arrow[d, hook] & (L\otimes\mathbb{R})^{\pm 1} \arrow[d, hook] \\
\mathrm{Stab}_{GL_2(\mathbb{R})}(f) \arrow[r, "\sim"] & (L\otimes\mathbb{R})^\times
\end{tikzcd}$$

$\mathrm{Cl}(L) \longleftrightarrow GL_2(\mathbb{Z}) \backslash \{ f \in V(\mathbb{Z}) \mid \text{disc}(f) = D\}$

$\{*\} \longleftrightarrow GL_2^{\pm 1}(\mathbb{R}) \backslash \{ f \in V(\mathbb{R}) \mid \text{disc}(f) = D\}$
[$\mathbb{R}^\times \subseteq GL_2(\mathbb{R})$ acts trivially]

$\{*\} \longleftrightarrow GL_2(\mathbb{R}) \backslash \{ f \in V(\mathbb{R}) \mid \text{disc}(f) = D\}$

$$(L\otimes\mathbb{R})^\times = \begin{cases} \mathbb{C}^\times \\ \mathbb{R}^\times\times\mathbb{R}^\times \end{cases}$$

Let $GL_2^{\pm 1}(\mathbb{R}) = \{ M \in GL_2(\mathbb{R}) \mid \det(M) = \pm 1\}$,

$$(L\otimes\mathbb{R})^{\pm 1} = \{ x \in (L\otimes\mathbb{R})^\times \mid \mathrm{Nm}(x) = \pm 1\} = \begin{cases} S^1 = \{x \in \mathbb{C}^\times \mid |x| = 1\} \\ \{(s,t) \in \mathbb{R}^\times\times\mathbb{R}^\times \mid st = \pm 1\} \cong \{\pm 1\}\times\mathbb{R}^\times \end{cases}$$

To prove this, we need a more general concept of fundamental domains. For motivation:

Lemma 8.10

Let G act transitively on X with finite stabilisers.
Let α be a fund. dom. for the left action of $H \subseteq G$ on G.
Let $x_0 \in X$. For any $x \in X$, let $\beta(x) = \sum_{\substack{g \in G: \\ x = g x_0}} \alpha(g)$.

Then, $\sum_{x \in Hy} \beta(x) = [\mathrm{Stab}_G(y) : \mathrm{Stab}_H(y)]$ for all $y \in \underset{G x_0}{\overset{\shortparallel}{X}}$.

Rmk If $\alpha = \mathbb{1}_A$ for some subset A of G, then β is the characteristic function of the multiset $A x_0$.

Rmk We'd like to apply this with $X = \{ f \in V(\mathbb{R}) \mid \mathrm{disc}(f) = D \}$, $G = GL_2(\mathbb{R})$, $H = GL_2($

Pf of Lemma

Let $y = a x_0$, $a \in G$.

$$\sum_{x \in Hy} \beta(x) \underset{x = hy}{=} \frac{1}{\# \mathrm{Stab}_H(y)} \sum_{h \in H} \beta(hy)$$

$$= \text{ — '' — } \sum_{h \in H} \sum_{\substack{g \in G: \\ hy = g x_0 \\ \shortparallel \\ h a x_0}} \alpha(g)$$

$$\underset{g = has}{=} \text{ — '' — } \sum_{h \in H} \sum_{s \in \mathrm{Stab}_G(x_0)} \alpha(has)$$

$$\underset{(\alpha \text{ f.d. for } H \subseteq G)}{=} \frac{\# \mathrm{Stab}_G(x_0)}{\# \mathrm{Stab}_H(y)} = \frac{\# \mathrm{Stab}_G(y)}{\# \mathrm{Stab}_H(y)}. \qquad \square$$

We need to work with infinite stabilisers, though!

<u>Lemma 8.11</u>

Let G act transitively on X.

Let α be a fund. dom. for the left action of a <u>countable</u> subgroup $H \leq G$ on G.

For each $x \in X$, assume $\mathrm{Stab}_G(x)$ is a topological group with Haar measure $d_x s$ and assume that they are compatible under the isomorphisms

$$\begin{array}{ccc} \mathrm{Stab}_G(x) & \xrightarrow{\sim} & \mathrm{Stab}_G(gx) \\ s & \longmapsto & gsg^{-1} \end{array}.$$

(i.e. this is a homeom. and $d_{gx}(gsg^{-1}) = d_x s$.)

Let $x_0 \in X$. For any $\underset{gx_0}{x} \in X$, let $\beta(x) = \int_{\mathrm{Stab}_G(x_0)} \alpha(gs)\, d_{x_0}s = \int_{\mathrm{Stab}_G(x)} \alpha(sg)\, d_x s$.

(This is independent of the choice of g!)

Then, $\sum_{x \in Hy} \beta(x) = \mathrm{vol}(\mathrm{Stab}_H(y) \backslash \mathrm{Stab}_G(y))$ for all $y \in X$,

assuming there is a measurable fund. dom. σ for the left action of $\mathrm{Stab}_H(y)$ on $\mathrm{Stab}_G(y)$.

<u>Rmk</u> The previous lemma is the case where $\mathrm{Stab}_G(x)$ has the discrete top. and counting measure (assuming H is countable)

Pf. Let $y = a x_0$, $a \in G$.

$$\sum_{x\in Hy} \beta(x) = \sum_{h\in H/\mathrm{Stab}_H(y)} \beta(\underbrace{hy}_{hax_0})$$

$$= \sum \int_{\mathrm{Stab}_G(hy)} \alpha(sha)\, d_{hy}s$$

$$\underset{s=hth^{-1}}{=} \sum \int_{\mathrm{Stab}_G(y)} \alpha(hta)\, d_y t$$

$$\underset{\substack{\sigma \text{ fund.} \\ \text{dom. for } \mathrm{Stab}_H(y) \curvearrowright \mathrm{Stab}_G(y)}}{=} \sum \int \sum_{v\in \mathrm{Stab}_H(y)} \sigma(v^{-1}t)\,\alpha(hta)\, d_y t$$

$$\underset{t=vs}{=} \sum \int \sum \sigma(s)\,\alpha(hvsa)\, d_y s$$

$$= \sum_{h\in H} \int \sigma(s)\,\alpha(hsa)\, d_y s$$

$$\underset{\substack{\alpha \text{ fund. dom.} \\ \text{for } H \curvearrowright G}}{=} \int \sigma(s)\, d_y s$$

$$= \mathrm{vol}\left(\mathrm{Stab}_H(y) \backslash \mathrm{Stab}_G(y)\right). \qquad \square$$

<u>If $D<0$</u> (imaginary case), then

$$vol\left(Stab_{GL_2(\mathbb{Z})} \backslash Stab_{GL_2^{\pm 1}(\mathbb{R})}\right) = \frac{1}{\# \mathcal{O}_L^\times} \cdot \underbrace{vol(S^1)}_{2\pi \ (?)} .$$

Taking $X := \{ f \in V(\mathbb{R}) \mid disc(f) = D \}$, $\alpha :=$ Minkowski's fund. dom (for $\|\cdot\|_2$),

$$x_0 := \frac{\sqrt{|D|}}{2}(X^2+Y^2) \in V(\mathbb{R})$$

$$Stab_{GL_2^{\pm 1}(\mathbb{R})}(x_0) = O_2(\mathbb{R}),$$

one gets $\beta \simeq c \cdot 1_{\{|b| \le a \le c\}}$ ~~as before~~

(as last time).

If $D>0$ (real case), then

$$\mathrm{vol}\left(\mathrm{Stab}_{GL_2(\mathcal{O})} \backslash \mathrm{Stab}_{GL_2^{\pm 1}(\mathbb{R})}\right) = R_K \quad \text{(times a constant)}$$

Take $x_0 = \sqrt{D}\, XY \in V(\mathbb{R})$.

$$\mathrm{Stab}_{GL_2^{\pm 1}(\mathbb{R})}(x_0) = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}$$

Cleaner example of fund. dom. α to use:

Lemma 8.12 The set A of matrices $g = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \in GL_2(\mathbb{R})$ such that

a) $x_2, y_1 \geq x_1, -y_2 \geq 0$

or

b) $x_2, -y_1 \geq x_1, y_2 \geq 0$

is an almost fundamental domain for $GL_2(\mathbb{Z}) \curvearrowright GL_2(\mathbb{R})$.

Let α be the corr. fund. dom. If the strict inequalities are satisfied, then $\alpha(g) = 1$.