Let K be a field with $\mathrm{char}(K) \neq 2$. For any $D \in K^\times$, consider the K-algebra $L_D = K[X]/(X^2 - D)$ of degree 2. Rmk If $D \notin K^{\times 2}$, then $L_D \cong K(\sqrt{D})$.
$\alpha_D \leftrightarrow \sqrt{D}$

Otherwise, $L_D \cong K \times K$.
$\alpha_D \leftrightarrow (\sqrt{D}, -\sqrt{D})$

Let $\alpha_D \in L_D$ be the image of X.

Rmk $(1, \alpha_D)$ and $(1, \tau_D)$ are K-bases of L_D.

$$\tau_D = \frac{D + \alpha_D}{2}$$

Lemma 8.2 Let $K = \mathbb{Q}$. If L is a quadr. number field ~~an étale ext. of $\mathbb{Q}$ of degree 2~~ with discriminant D, then $L \cong L_D$ and $(1, \tau_D)$ is a $\mathbb{Z}$-basis of the ring of integers of L_D.

Pf ~~The min. pol. of $a + b\tau_D$ is~~

$$\cancel{\left(X - a - b\cdot\frac{D+\alpha_D}{2}\right)\left(X - a - b\cdot\frac{D-\alpha_D}{2}\right)}$$

$$\cancel{= \left(X - a - b\cdot\frac{D}{2}\right)^2 - \left(b\cdot\frac{\alpha_D}{2}\right)^2}$$

$$\cancel{= X^2 + a^2 + b^2\frac{D^2}{4} - 2aX - bDX + abD - b^2\frac{D}{4}}$$

Let $L = \mathbb{Q}(\sqrt{t})$, $t \in \mathbb{Z}$ squarefree.

If $t \not\equiv 1 \bmod 4$, then $(1, \sqrt{t})$ is an integral basis and $D = 4t$

If $t \equiv 1 \bmod 4$, then $\left(1, \frac{1+\sqrt{t}}{2}\right)$ — " — $D = t$. □

Def Such a number $D \in \mathbb{Z}$ is a fundamental discriminant.

Rmk D is a fund. disc. if and only if

$D \neq 0, 1$ and: $D \equiv 1 \bmod 4$ is squarefree, or $\mathbb{Z} \ni \frac{D}{4} \not\equiv 1 \bmod 4$ is sqfree.

~~D is squarefree, and~~

~~$D \equiv 1$~~

Thm 8.3 We have a $GL_2(K)$-equivariant bijection

$$L_D^\times \backslash \{K\text{-basis } (\omega_1,\omega_2) \text{ of } L_D\} \xleftrightarrow{(*)} \{f \in V(K) \mid \operatorname{disc}(f) = D\}$$

$$[(\omega_1,\omega_2)] \longmapsto \frac{\mathrm{Nm}_{L_D|K}(X\omega_1 + Y\omega_2)}{\mathrm{Nm}(\omega_1,\omega_2)}$$

$$\text{with } \mathrm{Nm}(\omega_1,\omega_2) := \det_K\begin{pmatrix} 1 \mapsto \omega_1 \\ \tau_D \mapsto \omega_2 \end{pmatrix}$$

$$\left[\left(1, \frac{b+\alpha_D}{2a}\right)\right] \text{ if } a \neq 0 \longleftarrow\!\!\mapsto aX^2 + bXY + cY^2$$

$$\ldots \quad \text{if } a = 0$$

Here, $L_D^\times$ acts on {basis} by mult.: $s(\omega_1,\omega_2) = (s\omega_1, s\omega_2)$

And $GL_2(K)$ — " — by matrix mult., considering (ω_1,ω_2) a vector.

Pf The map $L_D^\times \backslash \{\text{basis}\} \to V(K)$ is well-def.:

If we apply $s \in L_D^\times$, then the numerator and denominator of the RHS are multiplied by $\mathrm{Nm}(s) = \det_K(\text{mult. by } s)$.

The map is $GL_2(K)$-equivariant:

The RHS can be defined by

$$f(v) = \frac{\mathrm{Nm}\left(\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} \cdot v\right)}{\det\begin{pmatrix} 1 \mapsto \omega_1 \\ \tau_D \mapsto \omega_2 \end{pmatrix}}.$$

(dot product)

$$\Rightarrow M(\omega_1,\omega_2) = (\omega_1',\omega_2') \text{ is sent to } \frac{\mathrm{Nm}\left(M\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} \cdot v\right)}{\det\begin{pmatrix} 1 \mapsto \omega_1' \\ \tau_D \mapsto \omega_2' \end{pmatrix}} = \frac{\mathrm{Nm}\left(\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} \cdot M^T v\right)}{\det\begin{pmatrix} 1 \mapsto \omega_1 \\ \tau_D \mapsto \omega_2 \end{pmatrix} \cdot \det(M)} = (Mf)(v)$$

We have $\operatorname{disc}(f) = D$:

Since $GL_2(K)$ acts transitively on {basis} and $\operatorname{disc}(Mf) = \operatorname{disc}(f)$, it suffices to check this for one basis $(1, \alpha_D)$, for which $f = \frac{X^2 - DY^2}{2}$.

That the maps are inverses can be checked directly. $\square$

Cor 8.4 a) $GL_2(K)$ acts transitively on $\{f \mid \mathrm{disc} = D\}$.

b) $\mathrm{Stab}_{GL_2(K)}(f) \cong L_D^\times$.

Pf a) $GL_2(K)$ acts transitively on {basis}.

b) ~~[illegible]~~ we have

$$\mathrm{Stab} \longrightarrow L_D^\times$$

$$M \longmapsto s \in L_D^\times \text{ such that } M(\omega_1, \omega_2) = s(\omega_1, \omega_2).$$

$\square$

Thm 8.5 Let $K = \mathbb{Q}$ and let L be a quadr. n.f. with discriminant D.

Then, (*) restricts to a $GL_2(\mathbb{Z})$-equivariant bijection

$$L_D^\times \backslash \{\mathbb{Z}\text{-basis } (\omega_1, \omega_2) \text{ of a fractional ideal of } L_D\} \longleftrightarrow \{f \in V(\mathbb{Z}) \mid \mathrm{disc}(f) = D\}$$

Pf Let $\mathfrak{a} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$.

"$\Rightarrow$" If $\mathfrak{a}$ is a fractional ideal, then

~~[illegible]~~

$$x\omega_1 + y\omega_2 \in \mathfrak{a} \quad \forall x, y \in \mathbb{Z}$$

$\Rightarrow N_m(x\omega_1 + y\omega_2)$ is divisible by $N_m(\mathfrak{a}) = |N_m(\omega_1, \omega_2)|$

$\Rightarrow f(x,y) \in \mathbb{Z} \quad \forall x, y \in \mathbb{Z}$

$\Rightarrow f \in V(\mathbb{Z})$.

"$\Leftarrow$" If $f \in V(\mathbb{Z})$, we can take $\omega_1 = 1$, $\omega_2 = \frac{b + \sqrt{D}}{2a}$.

(Note that $D = b^2 - 4ac \notin \mathbb{Q}^{\times 2}$, so $a \neq 0$.)

We need to check that $\underbrace{\mathbb{Z}[\tau_D]}_{=\mathcal{O}_L} \cdot \mathfrak{a} \subseteq \mathfrak{a}$, i.e. that $\tau_D \mathfrak{a} \subseteq \mathfrak{a}$:

$$\tau_D\omega_1 = \tau_D = \frac{D-b}{2} \cdot \omega_1 + a \cdot \omega_2 \in \mathfrak{a} \quad \text{because } D \equiv b^2 - 4ac \equiv b \bmod 2$$

$$\tau_D\omega_2 = -c \cdot \omega_1 + \frac{D+b}{2} \cdot \omega_2 \in \mathfrak{a}$$

$\square$

<u>Cor 8.6</u> a) We obtain a bijection

$$Cl(L) \longleftrightarrow GL_2(\mathbb{Z}) \backslash \{f \in V(\mathbb{Z}) \mid \mathrm{disc} = D\}.$$

b) $\mathrm{Stab}_{GL_2(\mathbb{Z})}(f) \cong \mathcal{O}_L^\times$.

<u>Pf</u> a) $GL_2(\mathbb{Z})$ acts transitively on the $\mathbb{Z}$-bases of $\mathfrak{a}$.

b) If $M(\omega_1, \omega_2) = s(\omega_1, \omega_2)$ with $M \in GL_2(\mathbb{Z})$, then $\mathfrak{a} = s\mathfrak{a}$, so $s \in \mathcal{O}_L^\times$.

If $s \in \mathcal{O}_L^\times$, then $\mathfrak{a} = s\mathfrak{a}$, so there is a change of $\mathbb{Z}$-basis sending (ω_1, ω_2) to $s(\omega_1, \omega_2)$. $\square$

~~Remark In particular, $GL_2(\mathbb{Z}) \circlearrowright \{f \in V(\mathbb{Z}) \mid \mathrm{disc} = D\}$ has a fund. dom. if and only if $\mathcal{O}_L^\times$~~

<u>Lemma 8.7</u> a) If $D > 0$, then $GL_2(\mathbb{Z}) \circlearrowright \{f \in V(\mathbb{Z}) \mid \mathrm{disc} = D\}$ has no fund. dom.

b) If $D < 0$, then ~~there is a~~

$S := \{f \in V(\mathbb{Z}) \mid \mathrm{disc} = D, \ |b| \leq a \leq c\}$

is an almost fund. dom. The associated fund. dom. χ satisfies $\chi(f) = \frac{1}{2}$ for all f in the interior of S.

Pf a) $\mathrm{Stab} \cong \mathcal{O}_L^\times$ is infinite.

b) We have an $SL_2(\mathbb{R})$-equivariant

$$GL_2(\mathbb{R})/O_2(\mathbb{R}) \longleftrightarrow \{ f \in V(\mathbb{R}) \mid \mathrm{disc}(f) < 0 \}$$

$$M = \begin{pmatrix} - v_1 - \\ - v_2 - \end{pmatrix} \longmapsto \| X v_1 + Y v_2 \|^2$$

with $-4\det(M) = \mathrm{disc}(f)$.

Minkowski's almost fund. dom. $\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mid \|v_1\| \le \|v_2\|,\ |v_1 \cdot v_2| \le \frac{1}{2}\|v_1\|^2 \}$ for $GL_2(\mathbb{Z}) \circlearrowright GL_2(\mathbb{R})/O_2(\mathbb{R})$ is mapped to $\{ f \mid |b| \le a \le c \}$.

We have $GL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) \cup \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} SL_2(\mathbb{Z})$.

Think about how $\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ acts on LHS and RHS...

"□"

Thm 8.8 $\displaystyle\sum_{\substack{K \text{ imag. quadr. n.f.} \\ 0 < -\mathrm{disc}(K) \le T}} h_K \sim C \cdot T^{3/2}$

~~We thus conclude~~

~~Ingredients~~

1) $\displaystyle\sum_{\substack{K \text{ imaginary} \\ \text{quadr. n. f.} \\ 0 < -\mathrm{disc}(K) \le T}} h_K$ $\cancel{\le \#\{ f \in V(\mathbb{Z}) \mid |b| \le a \le c,\ -(b^2 - 4ac) \le T \text{ fundamental disc.} \}}$

$$= \sum_{\substack{f \in V(\mathbb{Z}): \\ 0 < -\mathrm{disc}(f) \le T \\ \mathrm{disc}(f) \text{ fund. disc.}}} \chi(f) = \sum_{\substack{f \in V(\mathbb{Z}) \cap S \\ 4ac - b^2 \le T \\ -b^2 + 4ac \text{ fund. disc}}} \tfrac{1}{2} + O\Big(\sum_{\substack{f \in V(\mathbb{Z}) \cap \partial S \\ 4ac - b^2 \le T \\ -b^2 + 4ac \text{ f. d.}}} 1 \Big)$$

2) Cut off cusp: If $|a| < 1$, then $a = 0$, so $b = 0$, so $\mathrm{disc}(f) = b^2 - 4ac = 0$.

3) Davenport's lemma [Note: the region in $\mathbb{R}^3$ is scaled by a factor of $T^{1/2}$ $\Rightarrow \mathrm{vol} \sim T^{3/2}$]

4) Sieve for fund. disc.

(E. g. for any $p \neq 2$, remove $f \in V(\mathbb{Z})$ such that $p^2 \mid b^2 - 4ac$.)

Rmk: $\sum_{K \ldots} 1 \sim C'' \cdot T$ by Pset 1

$\rightsquigarrow$ on average, $h_K R_K$ is $\asymp T^{1/2}$ for a q.n.f. of disc $\asymp T$.

We will also ~~show~~ sketch how to show:

Thm 8.9

$$\sum_{\substack{K \text{ real q.n.f.} \\ 0 < \mathrm{disc}(K) \leq T}} h_K R_K \sim C' \cdot T^{3/2}$$

Brauer-Siegel Theorem

Let K be a n.f. of degree n and let $\varepsilon > 0$.

Then, $|D_K|^{\frac{1}{2}-\varepsilon} \ll_{n,\varepsilon} h_K R_K \ll_{n,\varepsilon} |D_K|^{\frac{1}{2}+\varepsilon}$

[These keep showing up together and are difficult to separate!]