$$\Rightarrow \tilde{N}(T) := \sum_{M \in SL_n(\mathbb{Z}) \backslash M_n^+(\mathbb{Z})} \tau(\det(M)/T)$$

$$= \sum_{M \in M_n^+(\mathbb{Z})} \tilde{f}_T(M)$$

$$\underset{\text{def. of } f * \eta}{=} \int_{SL_n(\mathbb{R})} f(h) \, S(T,h) \, d^\times h$$

~~with~~ $\cancel{S(g,T) :=} \sum_{M = \lambda g \in M_n^+(\mathbb{Z})} \underbrace{\tau(\lambda^n/T)\eta(h^{-1}g)}$,

~~smooth, cpt. supp.~~
~~fct. of M, scaled by~~
~~a factor of $T^{1/n}$ from $\tau(\lambda^n)\eta(h^{-1}g)$~~

with $S(T,h) := \sum_{M \in M_n^+(\mathbb{Z})} r_{h,T}(M)$,

$$r_{T,h}(\lambda g) = \tau(\lambda^n/T)\eta(h^{-1}g).$$

Note that $r_{T,h}(M) = r_{1,id}(T^{-1/n} h^{-1} M) = (T^{1/n}h) r_{1,id}(M)$ and that $r_{1,id}$ is smoo[th] and cpt. sup[p].
Assume f is Siegel's fund. dom., so $\mathrm{supp}(f) \subseteq S^{Siegel} = N' B' SO_n(\mathbb{R})$

Cutting off the cusp:

If $h = \underset{\substack{\cap \\ N'}}{n} \, \underset{\substack{\cap \\ B'}}{b} \, \underset{\substack{\cap \\ SO_n(\mathbb{R})}}{k}$, $T^{1/n} b = a \in A$, then any entry in the first row of a matrix in $\mathrm{supp}(r_{T,h}) = \underbrace{T^{1/n} h}_{\substack{nak_{\in SO_n(\mathbb{R})} \\ \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \begin{pmatrix} a_1 & \\ & \ddots \end{pmatrix}}} \underbrace{\mathrm{supp}(r_{1,id})}_{\text{compact}}$ (compact) has entries $\ll a_1$.

$\Rightarrow$ If $a_1 \ll 1$ (for a suff. small constant), then there is no $M \in M_n^+(\mathbb{Z}) \cap \mathrm{supp}(r_{T,h})$, so $S(T,h) = 0$. (All entries in the first row would have to be 0.) ~~[illegible]~~

Otherwise (if $a_1 \gg 1$), we apply Poisson summation.

$$S(T,h) = \sum_{M \in \underbrace{(T^{-1/n} h^{-1}) M_n^{+}(\mathbb{Z})}} r_{1,id}(M)$$

lattice $\Lambda = \underbrace{\Lambda' \oplus \dots \oplus \Lambda'}_{n \text{ (columns)}}$
with Λ' spanned by the
columns of $T^{-1/n} h^{-1}$

The ~~[illegible]~~ lattice Λ' is the dual of the lattice Λ'' spanned by the rows of $T^{1/n} h$. $\underset{\text{Rat}}{\overset{\uparrow}{\Longrightarrow}}$ The succ. min. of Λ' are the inverses of the succ. min. of Λ'', which are $\asymp a_1, \dots, a_n$ by Thm 7.4.1 b).

The succ. min of Λ are those of Λ', repeated n times.

$$\underset{\text{Thm 4.2.6}}{\Longrightarrow} \quad S(T,h) = T^n \int r_{1,id}(M)\, dM + O(T^n a_1^{-k})$$

$$= \int r_{T,h}(M)\, dM + O(T^n a_1^{-k})$$

$$\Rightarrow \hat{N}(T) = \int_{N' \cdot B' \cdot SO_n(\mathbb{R})} f(h) \Big(\underbrace{\int r_{T,h}(M)\, dM}_{\int_0^\infty \int_{SL_n(\mathbb{R})} \tau(\lambda^n/T)\, \underset{\eta(h^{-1}g)}{\lambda} \lambda^{n^2} u\, d^\times g\, d^\times \lambda} + O(T^n \underset{\sim}{a_1^{-k}}) \Big)\, d^\times h$$

as $a_1 \to \infty$ → 0

$$\left(\frac{T^{1/n}}{(b_1^{n-1} \cdots b_{n-1})^{1/n}} \asymp \right) a_1 \gg 1$$

↑
as $T \to \infty$, the set of such $h \in S^{\text{Siegel}}$ converges to S^{Siegel}

$$\| \quad T^n \int_{SL_n} \eta(h^{-1}g)\, d^\times g \cdot \int_0^\infty n\tau(\lambda^n) \lambda^{n^2} d^\times \lambda$$

$$= T^n \cdot \int_0^\infty n \tau(\lambda^n) \lambda^{n^2} d^\times \lambda \cdot \underbrace{\int (f * \eta)(g)\, d^\times g}_{\mathrm{vol}(\dots)} + o_{T \to \infty}(T^n).$$

Approximate $1_{[1,2]}$ from above and below by functions τ.

$\rightsquigarrow$ upper and lower bound for $N(2T)-N(T)$.

The result follows by taking the limit $T \to \infty$ for better and better approximations. $\square$

We can also compute volumes over $\mathbb{Z}_p$:

Thm 7.5.2 $\quad vol(GL_n(\mathbb{Z}_p)) = \prod_{i=1}^{n} (1-p^{-i})$.

Pf $\quad M \in M_n(\mathbb{Z}_p)$ lies in $GL_n(\mathbb{Z}_p)$ iff $(M \bmod p) \in GL_n(\mathbb{F}_p)$.

$$\mathbb{P}((M \bmod p) \in GL_n(\mathbb{F}_p)) = \prod_{i=1}^{n} \underbrace{\mathbb{P}(v_i \text{ lin. indep. from } v_1,\dots,v_{i-1} \text{ assuming } v_1,\dots,v_{i-1} \text{ are lin. indep.})}_{1-p^{(i-1)-n}}$$

Write $M \bmod p = \begin{pmatrix} -v_1- \\ \vdots \\ -v_n- \end{pmatrix}$

$\square$

Thm 7.5.3 $\quad vol(SL_n(\mathbb{Z}_p)) = \prod_{i=2}^{n} (1-p^{-i})$

Pf $\quad \mathbb{Z}_p^\times \times SL_n(\mathbb{Z}_p) \longleftrightarrow GL_n(\mathbb{Z}_p)$

$$(t, h) \longmapsto \begin{pmatrix} 1 & & \\ & \ddots & \\ & & t \end{pmatrix} h = g$$

$$d^\times t \, d^\times h = d^\times g$$

$$\Rightarrow \underbrace{vol(\mathbb{Z}_p^\times)}_{= 1-p^{-1}} \cdot vol(SL_n(\mathbb{Z}_p)) = \underbrace{vol(GL_n(\mathbb{Z}_p))}_{= \prod_{i=1}^{n}(1-p^{-i})}$$

$\square$

<u>Cor 7.5.4</u> $\mathrm{vol}(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})) \cdot \prod_p \mathrm{vol}(SL_n(\mathbb{Z}_p)) = 1.$

This is not a coincidence! It is closely related to the fact that $SL_n(\mathbb{Q})$ has Tamagawa number 1:

$$\mathrm{vol}(SL_n(\mathbb{Q})\backslash SL_n(\mathbb{A}(\mathbb{Q}))) = 1$$

8. Ideals in quadratic number fields

For any (comm.) ring R, let $V(R)$ be the set of binary quadr. forms with coeff. in R: $f(X,Y) = aX^2 + bXY + cY^2$ $(a,b,c \in R)$

~~The discriminant~~

Let $GL_2(R)$ act on $V(R)$ by

$$(Mf)(v) = \det(M)^{-1} \cdot f(M^T v) \quad \text{for } M \in GL_2(R),\ f \in V(R),\ v \in R^2.$$

Lemma 8.1 The discriminant $\operatorname{disc}(f) = b^2 - 4ac$ is an invariant:

$$\operatorname{disc}(Mf) = \operatorname{disc}(f).$$

Remark $\begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} f = f$, so the action factors through $PGL_2(R) = GL_2(R)/R^\times$