a) "$<\infty$ el. of each orbit": ~~[crossed out]~~ There are only finitely many $v_i \in \Lambda$ with $|v_i| \le a_i \simeq \lambda_i(\Lambda)$.

"≥ 1 el. of each orbit":

Construct $v_1, \dots, v_n$ inductively. To construct v_i:

Let $\tau_i : \mathbb{R}^n \to \mathbb{R}^n$ be the proj. onto the orth. complement of $\langle v_1, \dots, v_{i-1} \rangle$.

$\tau_i(\Lambda)$ is a lattice of rank $n-(i-1)$.

Choose $v_i \in \Lambda$ so that $a_i = \tau_i(v_i) \neq 0$ has minimal length and $|n_{ij}| \le \frac{1}{2}$ $\forall j < i$. (This can be arranged by adding integer multiples of $v_1, \dots, v_{i-1}$ to v_i.)

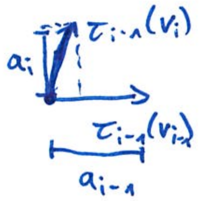
~~If we had a~~

~~Note that~~

If we had $a_i < \frac{\sqrt{3}}{2} a_{i-1}$, then

$$|\tau_{i-1}(v_i)|^2 < \left(\tfrac{1}{2} a_{i-1}\right)^2 + \left(\tfrac{\sqrt{3}}{2} a_{i-1}\right)^2 = a_{i-1}^2,$$

contradicting the minimality of $a_{i-1} = |\tau_{i-1}(v_{i-1})|$.



$\square$

7.5. Fundamental volume

Rmk W.r.t. Haar measure, $\mathrm{vol}(GL_n(\mathbb{Z})\backslash GL_n(\mathbb{R})) = \infty$.

Ex ($n=1$) $GL_1(\mathbb{Z})\backslash GL_1(\mathbb{R}) = \{\pm 1\}\backslash \mathbb{R}^\times = \mathbb{R}_{>0}$. $\mathrm{vol}(\ldots) = \int_0^\infty x^{-1}dx = \infty$.

Thm 7.5.1 With respect to the Haar measure on $SL_n(\mathbb{R})$,

$$\mathrm{vol}(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})) = \zeta(2)\zeta(3)\cdots\zeta(n).$$

Ex ($n=1$) $SL_1(\mathbb{R}) = \{1\}$ and the Haar measure is 1.

Ex ($n=2$) can be shown using the explicit Minkowski set by integrating.

Rmk Let $B' = \{(b_i)_i \in B \mid b_i \geq \frac{\sqrt{3}}{2}\ \forall i\}$

$$\mathrm{vol}(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})) \leq \mathrm{vol}(S^{\text{Siegel}} \cap SL_n(\mathbb{R}))$$

$$= \mathrm{vol}(N' \cdot B' \cdot SO_n(\mathbb{R}))$$

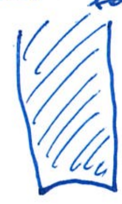
$$= \int_{N' \times B' \times SO_n(\mathbb{R})} \prod_{i>j} dn_{ij} \prod_i b_i^{-i(n-i)}\, d^\times b_i\, d^\times k$$

$$= \prod_{i>j} \underbrace{\int_{-1/2}^{1/2} dn_{ij}}_{1} \cdot \prod_{i=1}^{n-1} \underbrace{\int_{\sqrt{3}/2}^{\infty} b_i^{-i(n-i)} d^\times b_i}_{\left.\frac{b_i^{-i(n-i)}}{-i(n-i)}\right|_{b_i=\frac{\sqrt{3}}{2}}^{\infty} < \infty} \cdot \underbrace{\int_{SO_n(\mathbb{R})} d^\times k}_{<\infty}$$

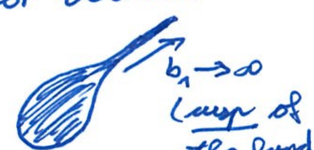
$<\infty$ because $SO_n(\mathbb{R})$ is compact

$< \infty$.

fund. dom. for SL_2



Better ~~picture~~ illustration for volume:



Pf of Thm. [Idea: approximate volume of fund. dom using "known" number of orbits.]

We estimate $N(T) := \# SL_n(\mathbb{Z}) \backslash \{ M \in M_{n\times n}(\mathbb{Z}) \mid 0 < \det(M) \leq T \}$.

in two ways:

a) ~~directly~~

b) using the volume of a fund. dom.

$\boxed{a)}$ Let $M^+_{n\times n}(\mathbb{Z}) = \{ M \in M_{n\times n}(\mathbb{Z}) \mid 0 < \det(M) \}$.
Every $SL_n(\mathbb{Z})$-orbit in $M^+_{n\times n}(\mathbb{Z})$ contains exactly one matrix M in Hermite normal form:

$$M = \begin{pmatrix} a_1 & b_{1,2} & \cdots & b_{1,n} \\ & a_2 & \ddots & \vdots \\ & & \ddots & b_{n-1,n} \\ 0 & & & a_n \end{pmatrix} \quad \text{with } a_1, \dots, a_n \geq 1 \text{ and } 0 \leq b_{ij} < a_j \ \forall i < j.$$

[You can use row transf. to put any matrix in this form: Go column by column. In column i, while two entries in rows $j_1, j_2 \geq i$ are $\neq 0$ (w.l.o.g. both >0), subtract the smaller from the larger. Make the remaining entry >0. Then, add multiples of row i to rows $1, \dots, i-1$.]

$$\Rightarrow N(T) = \sum_{\substack{a_1, \dots, a_n \geq 1: \\ a_1 \cdots a_n \leq T}} \underset{\#\{b_{1,2}\}}{\underset{\uparrow}{a_2}} \ \underset{\#\{(b_{13}, b_{23})\}}{\underset{\uparrow}{a_3^2}} \cdots a_n^{n-1}$$

$$= \sum_{m \leq T} \Big(m^{-s}\text{-coeff. in } \underbrace{\zeta(s)\zeta(s-1)\cdots\zeta(s-(n-1))}_{\substack{\text{rightmost pole at } s=n \\ \text{with residue } \zeta(2)\cdots\zeta(n)}} \Big)$$

$$\underset{\substack{\text{Wiener-Ikehara} \\ \text{(or compute}}}{\underset{\uparrow}{\sim}} \frac{1}{n} \zeta(2) \cdots \zeta(n) \cdot T^n$$

b) Let f be a measurable fund. dom. for $SL_n(\mathbb{Z}) \curvearrowright SL_n(\mathbb{R})$.
[Any $M \in M_n^+(\mathbb{R})$ can be written uniquely as λg with $\lambda \in \mathbb{R}_{>0}$, $g \in SL_n(\mathbb{R})$.]
Let $f_T(\underbrace{\lambda g}_{\in M_n^+(\mathbb{R})}) = 1_{(0,T]}(\lambda^n) \cdot f(g)$ for $\lambda \in \mathbb{R}_{>0}$, $g \in SL_n(\mathbb{R})$.

$$\Rightarrow N(T) = \sum_{M \in M_n^+(\mathbb{Z})} f_T(M)$$

$\Rightarrow$ Naively, $N(T) \sim \displaystyle\int_{M_n^+(\mathbb{R}) \subset GL_n(\mathbb{R})} f_T(M) \underbrace{d^+M}_{\det(M)^n \cdot d^\times M}$

$$\underset{\substack{\uparrow \\ \text{def. of } d^\times g}}{=} \int_{\mathbb{R}_{>0}} \int_{SL_n(\mathbb{R})} f_T(\lambda g) \cdot \lambda^{n^2} \cdot n \, d^\times g \, d^\times \lambda$$

$$= \underbrace{\int_0^{T^{1/n}} n \lambda^{n^2} \underbrace{d^\times \lambda}_{\frac{d\lambda}{\lambda}}}_{\left[\frac{1}{n}\lambda^{n^2}\right]_{\lambda=0}^{T^{1/n}}} \cdot \underbrace{\int_{SL_n(\mathbb{R})} f(g) \, d^\times g}_{vol(SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R}))}$$

$$= \frac{1}{n} vol(\ldots) \cdot T^n.$$

Combining with a), the result would follow.

One way ~~to~~ make the estimate rigorous:

Thickening the cusp

~~Replace f by~~ Smoothen f by replacing it by $f * \eta$ for $\eta : SL_n(\mathbb{R}) \to \mathbb{R}_{\geq 0}$ smooth and compactly supported with $\int \eta(g) d^\times g = 1$.

~~Replace $1_{(0,T]}$ by a smooth approximation $\tau : \mathbb{R}_{>0} \to$ (lower or upper bound) compactly supported~~

Let $\tau : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ be smooth and compactly supported.

Let $\tilde{f}_T(\lambda g) = \tau(\lambda^n / T) \cdot (f * \eta)(g)$.