7. GL_n and SL_n

7.1. Haar measures

Prop 7.1.1
Any locally cpt. Hausdorff ~~top.~~ group G has a left Haar measure $d_l g$ and a right Haar measure $d_r g$, unique up to mult. by a number in $\mathbb{R}_{>0}$.

Def G is unimodular if we can take $d_l g = d_r g = dg$. Then, dg is called a Haar measure on G.

Rmk We can take $d_r g = d_l(g^{-1})$.

Ex $K, K^\times, K^n, GL_n(K), SL_n(K)$ for any local field K.

We'll use the following Haar measures:

Ex Lebesgue measure $dx = d^+x$ on $\mathbb{R}$ $\qquad d(t+x) = dx$

$d^\times x = |x|^{-1} dx$ on $\mathbb{R}^\times \qquad d^\times(tx) = |tx|^{-1} d(tx) = d^\times x$

Ex

Ex K nonarch. loc. field with prime ideal $\mathfrak{q}_K$, residue field $\mathbb{F}_{q_K}$, normalised valuation v_K, norm $|x|_K = q_K^{-v_K(x)}$.
Normalize the Haar measure dx on K so that $\mathrm{vol}(\mathcal{O}_K) := \int_{\mathcal{O}_K} dx = 1$.

Rmk For any subset A of $\mathcal{O}_K/\mathfrak{q}_K^n$,

$$\mathrm{vol}(\{x \in \mathcal{O}_K \mid (x \bmod \mathfrak{q}_K^n) \in A\}) = \mathbb{P}_{x \in \mathcal{O}_K/\mathfrak{q}_K^n}(x \in A).$$

Rmk $d(tx) = |t|_K \, dx$ for $t \in K$.

Ex $d^\times x = |x|_K^{-1} dx$ on $K^\times$

Ex If da, db are our Haar measures on A, B, use the prod. measure on $A \times B$.

Let K be any local field.

Ex The Lebesgue measure d^+g on $GL_n(K) \subset M_n(K)$ is not a (mult.) Haar measure:

$$d^+(ag) = |\det(a)|_K^n \, d^+g \quad \text{for } a \in GL_n(K).$$

Left mult. by a on a column has determinant det(a). There are n columns.

$\Rightarrow$ $d^\times g = |\det g|_K^{-n} \, d^+g$ is a Haar measure

Ex The map $K^\times \times SL_n(K) \longrightarrow GL_n(K)$

$$(t, h) \longmapsto \begin{pmatrix} 1 & & \\ & \ddots & \\ & & t \end{pmatrix} h = th = g$$

is a homeomorphism (in fact a diffeomorphism). We normalize the Haar measure $d^\times h$ on $SL_n(K)$ so that $d^\times t \, d^\times h$ is the pull-back of $d^\times g$.

~~This is possible by~~

~~Lemma 7.1.2 Let G be~~

The pull-back must be left invariant because $d^\times g$ is a Haar measure!

Remark $\mathbb{R}_{>0} \times SL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is a homeom. and isom.

$$(\lambda, h) \mapsto \lambda h$$

The pull-back of $d^\times g$ is $n \, d^\times \lambda \, d^\times h$.

7.2. Minkowski sets ~~Recall: Elements of~~

~~Thm 7.2.1 S^{Mink}~~

Recall: Elements of $GL_n(\mathbb{R})$ corr. to bases $(b_1,\dots,b_n)$ of $\mathbb{R}^n$.

Two matrices lie in the same $GL_n(\mathbb{Z})$-orbit iff their bases span the same lattice.

~~$GL_n(\mathbb{R})$~~

⇝ An almost fund. dom. for $GL_n(\mathbb{Z}) \curvearrowright GL_n(\mathbb{R})$ corr. to an almost unique choice of basis of each full lattice $\Lambda \subseteq \mathbb{R}^n$.

Def ~~Thm 7.2.1~~ The <u>Minkowski set</u> S^{Mink} is the set of matrices $\begin{pmatrix} - b_1 - \\ \vdots \\ - b_n - \end{pmatrix} \in GL_n(\mathbb{R})$ so that $(b_1,\dots,b_n)$ is a directional basis for the lattice Λ spanned by $b_1,\dots,b_n$.

Thm 7.2.1 S^{Mink} is a measurable almost fund. dom. for $GL_n(\mathbb{Z}) \curvearrowright GL_n(\mathbb{R})$

Pf "≥ 1 el. of each orbit": clear

"$<\infty$ — " — ": There are only fin. many $b_i \in \Lambda$ with $|b_i| = \lambda_i$. □

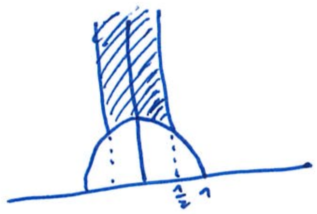
Rmk S^{Mink} is $\mathbb{R}^\times$-invariant and right $O_n(\mathbb{R})$-invariant

Exe ($n=2$, Euclidean norm) We have a bij.

$$\mathbb{R}^\times \backslash GL_2(\mathbb{R}) / O_2(\mathbb{R}) \longleftrightarrow \mathbb{H} = \{(x,y) \mid x \in \mathbb{R}, y \in \mathbb{R}_{>0}\}$$

$$\begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} \longleftarrow\!\shortmid (x,y)$$

The image of S^{Mink} is:



$$\boxed{\begin{pmatrix} -v_1- \\ -v_2- \end{pmatrix} \in S^{\text{Mink}} \Leftrightarrow |v_1| \le |v_2| \text{ and } |v_1 \cdot v_2| \le \frac{1}{2}|v_1|^2}$$

Rmk S^{Mink} is close to a fund. dom.: almost all lattices have exactly 2^n dir. bases (choices of signs of $\pm b_1, \dots, \pm b_n$).

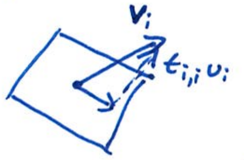
But it is difficult to check whether $M \in S^{\text{Mink}}$.

7.3. Iwasawa decomposition

Given a basis $(v_1,\dots,v_n)$ of $\mathbb{R}^n$, the Gram-Schmidt process produces the orth. basis $(u_1,\dots,u_n)$ s.t.

$$v_i = t_{i,1}u_1 + \dots + t_{i,i}u_i \quad \text{with } t_{ij}\in\mathbb{R},\ t_{ii}>0.$$

Here, $v_i - t_{i,i}u_i$ is the orth. proj. of v_i onto the subspace $\langle v_1,\dots,v_{i-1}\rangle = \langle u_1,\dots,u_{i-1}\rangle$. t_{ii} is the length of the perpendicular vector $t_{ii}u_i$.



Let $T := \left\{ \begin{pmatrix} * & 0 & 0 \\ \vdots & \ddots & 0 \\ * & \cdots & * \end{pmatrix} \in GL_n(\mathbb{R}) \text{ lower triangular with positive entries on the diagonal} \right\}.$

Lemma 7.3.1

a) $T \times O_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$ is a homeomorphism.
$\quad (t,k) \longmapsto tk$

b) The pullback of $d^\times g$ is ~~a scalar multiple of~~ $d_l^\times t\, d^\times k$, for a left Haar measure $d_l^\times t$ on T and a (right) Haar measure $d^\times k$ on $O_n(\mathbb{R})$.

Remark: $O_n(\mathbb{R})$ is unimodular.

Pf. a) follows from Gram-Schmidt process

b) The pull-back is left T and right $O_n(\mathbb{R})$-invariant. (The pull-back along $(t,k)\mapsto tk^{-1}$ is left $T\times O_n(\mathbb{R})$-invariant, hence a multiple of $d_l^\times t\, d_l^\times k = d_l^\times t\, d_r^\times k^{-1}$.) $\square$

Let $N := \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \right\} \subset J$ be the group of lower triangular unipotent matrices. Lemma 7.3.2 $\prod_{i>j} dn_{ij}$ is a (left) Haar measure on N.
~~Pf left mult by~~ Pf HW □

Let $A := \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \subset J$ be the group of diagonal matrices with positive entries on the diagonal. Write $a_1, \dots, a_n$ for the diagonal entries of $a \in A$.

Lemma 7.3.3

a) $N \times A \longrightarrow J$ is a homeom.
$(n, a) \longmapsto na$

b) The pullback of $d_l^\times t$ is a scalar multiple of

$$\prod_{i>j} \frac{a_j}{a_i} \, dn_{ij} \cdot \prod_i d^\times a_i = \prod_{i>j} dn_{ij} \cdot \prod_i a_i^{n+1-2i} \, d^\times a_i \qquad (\mathrm{I})$$

Pf a) is clear

b) ~~$\prod_{i>j} dn_{ij}$ is a left Haar measure on N:~~

~~Left mult. by n acts by a lower triangular unipotent matrix on the i-th column vector of the tangent space of N.~~ ~~$\mathbb{R}^{n-i}$~~

The measure (I) is left N-invariant by Lemma 7.3.2.

For left A-invariance, note that for $t \in A$,

$t n a = n' a'$ with $n'_{ij} = \frac{t_i}{t_j} n_{ij}$, $a' = ta$. □

Together:

Thm 7.3.4 (Iwasawa decomposition of $GL_n(\mathbb{R})$)

a) $$\begin{aligned} N \times A \times O_n(\mathbb{R}) &\longrightarrow GL_n(\mathbb{R}) \text{ is a diffeomorphism.} \\ (n, a, k) &\longmapsto nak \end{aligned}$$

b) $\prod_{i>j} \frac{a_j}{a_i} du_{ij} \prod_i d^\times a_i \, d^\times k$ is the pullback of a Haar measure on $GL_n(\mathbb{R})$.

Cor 7.3.5 (Iwasawa decomp. of $SL_n(\mathbb{R})$)

Let $B = \{ a \in A \mid \underset{\substack{\| \\ a_1 \cdots a_n}}{\det(A)} = 1 \} \cong (\mathbb{R}^{>0})^{n-1}$

$$a \longleftrightarrow (b_i)_{i=1,\dots,n-1} \text{ with } b_i = \frac{a_{i+1}}{a_i}$$

$$a_i = \frac{b_1 \cdots b_{i-1}}{(b_1^{n-1} \cdots b_{n-2}^2 \, b_1)^{1/n}}$$

a) $N \times B \times SO_n(\mathbb{R}) \longrightarrow SL_n(\mathbb{R})$ is a diffeom.

b) ~~The Haar measure~~

$\prod_{i>j} d n_{ij} \prod_i b_i^{-i(n-i)} d^\times b_i \, d^\times k$ is the pullback of a Haar measure

on $SL_n(\mathbb{R})$.

7.4. Siegel sets

Def Let $N' = \{n \in N \mid |n_{ij}| \le \frac{1}{2} \ \forall i > j\}$, $A' = \{a \in A \mid a_{i+1} \ge \frac{\sqrt{3}}{2} a_i \ \forall 1 \le i \le n-1\}$

The Siegel set S^{Siegel} is $N' \cdot A' \cdot O_n(\mathbb{R}) \subseteq GL_n(\mathbb{R})$

Thm 7.4.1 a) S^{Siegel} is a measurable almost fund. dom. for $GL_n(\mathbb{Z}) \hookrightarrow GL_n(\mathbb{R})$.

b) If $nak \in N'A'O_n(\mathbb{R})$ corr. to the lattice Λ, then

$$a_i \underset{\text{dim. } n, \, l.l}{\asymp} \lambda_i(\Lambda) \quad \text{for } i = 1, \dots, n.$$

Pf b) Let $nak = \begin{pmatrix} - v_1 - \\ \vdots \\ - v_n - \end{pmatrix}$, $k = \begin{pmatrix} - u_1 - \\ \vdots \\ - u_n - \end{pmatrix}$.

$$v_i = \sum_{j<i} n_{ij} a_j u_j + a_i u_i .$$

$$\Rightarrow |v_i| \le \sum_{j<i} \underbrace{|n_{ij}|}_{\substack{\le \frac{1}{2} \\ (n \in N')}} \cdot \underbrace{a_j}_{\substack{\ll a_i \\ (a \in A')}} + \{a_i\} \ll a_i$$

$$\Rightarrow \lambda_i \ll a_i$$

On the other hand,

$$a_1 \cdots a_n = |\det(nak)| = \operatorname{covol}(\Lambda) \underset{\substack{\uparrow \\ \text{Minkowski's} \\ \text{second theorem}}}{\asymp} \lambda_1 \cdots \lambda_n .$$