We can smoothen fund. dom.:

anything * smooth = smooth

fund. dom. * vol. 1 = fund. dom.

<u>Lemma 5.4</u> Let G be a locally compact Hausdorff group with left Haar measure $d_\ell g$ and right Haar measure $d_r g = d_\ell g^{-1}$.

Let $H \le G$ a subgroup and $f \in L^1(G)$ an integrable fund. dom. for $H \curvearrowright G$ (left mult.).

Let $\eta \in L^1(G)$ with $\int_G \eta(g)\, d_r g = 1$.

Then,

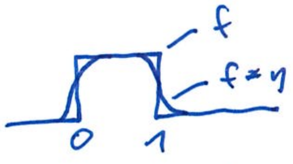
$$\cancel{(f*\eta)(g) = \int_G f(t)\cdot \eta(t^{-1}g)\, dt}$$

$$(f*\eta)(a) = \int_G f(b)\cdot \eta(b^{-1}a)\, d_\ell b \underset{\boxed{b=ac^{-1}}}{=} \int_G f(ac^{-1})\cdot \eta(c)\, d_r c$$

is also a fund. dom. for $H \curvearrowright G$.

Intuition: $f*\eta = \int_G \underbrace{f(b)\cdot b\eta}_{\substack{\text{left } b \text{ translate of } \eta \\ \text{(hopefully easy to count lattice points in here)}}} d_\ell b = \int_G \underbrace{fc}_{\substack{\text{right } c \text{ translate of } f \\ \text{(also a fund. dom.)}}} \cdot\, \eta(c)\, d_r c.$

<u>Ex</u> $\mathbb{Z} \curvearrowright \mathbb{R}$



<u>Pf</u> $\sum_{h\in H} (f*\eta)(ha) = \int_G \underbrace{\sum_{h\in H} f(hac^{-1})}_{1}\, \eta(c)\, d_r c = \int_G \eta(c)\, d_r c = 1.$ $\square$

6. The class number formula

~~Reminder~~

Let K be a number field of degree n and signature (r_1, r_2).

$$K \hookrightarrow \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$$

Define the hom. $\ell : \mathbb{R}^\times \longrightarrow \mathbb{R}$, $x \longmapsto \log|x|$

and $\ell : \mathbb{C}^\times \longrightarrow \mathbb{R}$, $x \longmapsto 2\log|x| = \log(x\bar{x})$

combine them to $\ell : (\mathbb{R}^{r_1} \times \mathbb{C}^{r_2})^\times \longrightarrow \mathbb{R}^{r_1+r_2}$.

Let $s : \mathbb{R}^{r_1+r_2} \longrightarrow \mathbb{R}$, $(t_1, \dots, t_{r_1+r_2}) \longmapsto t_1 + \dots + t_{r_1+r_2}$.

<u>Remark</u> $s(\ell(x)) = \log|\mathrm{Nm}(x)|$

In particular, $\ell(\mathcal{O}_K^\times) \subseteq H := \ker(s)$.

~~Reminder~~

We identify $H \xrightarrow{\sim} \mathbb{R}^{r_1+r_2-1}$, $(t_1, \dots, t_{r_1+r_2}) \longmapsto (t_1, \dots, t_{r_1+r_2-1})$.

[This defines a measure on H!]

<u>Reminder</u> The kernel of $\ell : \mathcal{O}_K^\times \longrightarrow H$ is the group μ_K of roots of unity in K.

The image is a full lattice in H, whose covolume is the <u>regulator</u> R_K.

Let $C \subseteq H$ be a fundamental cell.

Lemma 6.1 For any proj. $\pi : \mathbb{R}^{r_1+r_2} \twoheadrightarrow H$,

$$f(x) = \frac{1}{\#\mu_K} \cdot 1_C(\pi(L(x)))$$

is a fund. dom. for $\mathcal{O}_K^\times \curvearrowright_{\text{mult.}} (\mathbb{R}^{r_1} \times \mathbb{C}^{r_2})^\times$.

Pf $\displaystyle\sum_{u \in \mathcal{O}_K^\times} f(ux) = \sum_{v \in L(\mathcal{O}_K^\times)} \frac{\#\mu_K}{\#\mu_K} \cdot 1_C(v + \pi(L(x))) = 1$

C is fund. cell for $L(\mathcal{O}_K^\times) \subseteq H$

$\displaystyle\sum_{u \in \mathcal{O}_K^\times} \frac{1}{\#\mu_K} \cdot 1_C(\pi(L(u)) + \pi(L(x)))$

□

Lemma 6.2 Let $S(T) = \{ a \in (\mathbb{R}^{r_1} \times \mathbb{C}^{r_2})^\times \mid |\mathrm{Nm}(a)| \le T \}$.

Then, $f_T(x) := f(x) \cdot 1_{S(T)}(x)$ is a fund. dom. for $\mathcal{O}_K^\times \curvearrowright S(T)$

with volume $\displaystyle\int_{S(T)} f(x)\,dx = \frac{2^{r_1}(2\pi)^{r_2} R_K}{\#\mu_K} \cdot T$.

Pf $f_T(x) = f_1(x/T^{1/n})$. $\Rightarrow \int f_T = (T^{1/n})^n \cdot \int f_1 = T \cdot \int f_1$.

$\Rightarrow$ It suffices to consider $T = 1$.

All fund. dom. have same vol. $\rightsquigarrow$ w.l.o.g., π is the proj. ignoring the last coord.

We perform a change of variables on $(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2})^\times$:

Write elements of $\mathbb{R}^\times$ as $x = \pm e^z$ $(z \in \mathbb{R})$

and elements of $\mathbb{C}^\times$ as $x = e^{\frac{z}{2}+it}$ $(z \in \mathbb{R},\ 0 \le t < 2\pi)$.

Let $E = \{ z \in \mathbb{R}^{r_1+r_2} \mid s(z) \le 0 \}$.

Then,

$$\int_{S(T)} f(x)\,dx = \frac{2^{r_1}\pi^{r_2}}{\#\mu_K}\cdot\int_{E\cap\pi^{-1}(C)} \exp(z_1+\dots+z_{r_1+r_2})\,dz$$

$$\underset{z}{=} \frac{2^{r_1}\pi^{r_2}}{\#\mu_K}\cdot\int_C\int_{-\infty}^{0} \exp(r)\,dr\;d(z_1,\dots,z_{r_1+r_2-1})$$

$C\subseteq H\cong\mathbb{R}^{r_1+r_2-1}$

$$= \frac{2^{r_1}\pi^{r_2}R_K}{\#\mu_K}.$$

□

Thm 6.3 $\#\{\underset{\neq 0}{\mathfrak{a}}\subseteq\mathcal{O}_K$ principal ideal $\mid \mathrm{Nm}(\mathfrak{a})\le T\} \underset{K}{\sim} \dfrac{2^{r_1}(2\pi)^{r_2}R_K T}{\#\mu_K|\mathrm{disc}(K)|^{1/2}}$

for $T\to\infty$.

Pf $\mathrm{LHS} = \#\,\mathcal{O}_K^\times\backslash(\mathcal{O}_K\cap S(T))$

$$= \sum_{x\in\mathcal{O}_K} f_T(x)$$

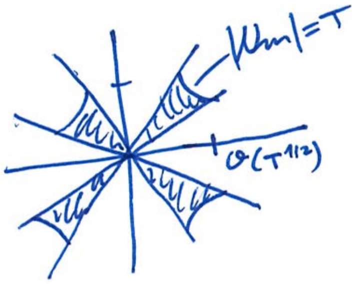
If we use the projection $\pi(z) = z - \frac{s(z)}{n}\cdot(\underbrace{1,\dots,1}_{r_1},\underbrace{2,\dots,2}_{r_2})$,

then $f(\lambda x) = f(x)\ \forall\lambda\in\mathbb{R}^\times$.

Rmk For $K=\mathbb{Q}(i)$, this is $\{x+iy\in\mathbb{Z}[i]\mid x^2+y^2\le T\}$
($\rightsquigarrow$ Gauß circle problem)

We can use Davenport's lemma (after replacing by a semialgebraic approximation!)

$$\Rightarrow LHS = \frac{\int f_T(x)\,dx}{\mathrm{covol}(\mathcal{O}_K)} + \mathcal{O}\left(T^{n-1}\right) \text{ for } T \to \infty.$$



the proj. of supp(f_T) onto each axis in $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \cong \mathbb{R}^n$ has length $\mathcal{O}(T^{1/n})$.

□

Pf 2 Using lemma 5.4, replace $1_C : H \to \{0,1\}$ by $1_C \underset{H}{*} \eta$ for a smooth compactly supported function $\eta : H \to \mathbb{R}_{\geq 0}$ with $\int_H \eta(z)\,dz = 1$.

Also, replace $1_{[0,T]}(|Nm(x)|) = 1_{S(T)}(x)$ by a smooth compactly supported approximation $\tau(|Nm(x)|/T)$.

$$\sum_{\substack{0 \neq \mathfrak{a} \subseteq \mathcal{O}_K \\ \text{principal}}} \tau(\mathfrak{a}) = \sum_{x \in \mathcal{O}_K} \frac{1}{\#\mu_K} (1_C * \eta)(\pi(l(x))) \cdot \tau(Nm(x)/T)$$

$$= \int_H \frac{1_C(h)}{\#\mu_K} \underbrace{\sum_{x \in \mathcal{O}_K} \eta(\pi(l(x)) - h)\, \tau(|Nm(x)|/T)}_{\substack{\text{smooth function of } x; \\ \text{the parameter } T \text{ scales the fct.} \\ \text{by a factor of } T^{1/n}}} dh$$

$$= \int_H \frac{1_C(h)}{\#\mu_k \cdot \mathrm{covol}(\mathcal{O}_k)} \Big[\int_{\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}} \eta(\pi(L(x)) - h)\, \tau(|N_m(x)|/T)\, dx + \mathcal{O}_k(T^{-k}) \Big] dh$$

Thm 4.2.6

$$= \frac{1}{\mathrm{covol}(\mathcal{O}_k)} \int_{\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}} \underbrace{\frac{(1_C * \eta)(\pi(L(x)))}{\#\mu_k}}_{\substack{(\text{fund. dom.})(x) \\ \text{for } \mathcal{O}_k^\times \circlearrowright \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}}} \underbrace{\tau(|N_m(x)|)}_{\substack{\text{approx.} \\ \text{to } 1_{(0,1]}(|N_m(x)|) \\ = 1_{S(1)}(x)}} dx \cdot T + \mathcal{O}_k(T^{-k})$$

As you let τ go to $1_{(0,1]}$, (say monotonely ptwise a.e.) ~~this goes to~~ the integral goes to $\int f_T(x)\,dx$ by monotone convergence. $\square$

Thm 6.4 (class number formula)

$$\#\{0 \neq \mathfrak{a} \in \mathcal{O}_K \text{ ideal} \mid \mathrm{Nm}(\mathfrak{a}) \leq T\} \sim \frac{2^{r_1}(2\pi)^{r_2} R_K h_K}{\#\mu_K |\mathrm{disc}(K)|^{1/2}} \cdot T$$

Pf Consider an ideal class $c \in \mathcal{Cl}(K)$. Let $\mathfrak{b} \in c$ be a fractional ideal.

$$\{0 \neq \mathfrak{a} \text{ ideal in } c\} \longleftrightarrow \mathcal{O}_K^\times \backslash \mathfrak{b}^{-1}$$

$$x \cdot \mathfrak{b} \longleftarrow\!\!\shortmid \; x$$

$$\mathrm{Nm}(\mathfrak{a}) \quad = |\mathrm{Nm}(x)| \cdot \mathrm{Nm}(\mathfrak{b}).$$

As in the prev. Thm.,

$$\#\left(\mathcal{O}_K^\times \backslash \{x \in \mathfrak{b}^{-1} \mid |\mathrm{Nm}(x)| \leq \tfrac{T}{\mathrm{Nm}(\mathfrak{b})}\}\right)$$

$$\sim \frac{2^{r_1}(2\pi)^{r_2} R_K}{\#\mu_K \underbrace{\mathrm{covol}(\mathfrak{b}^{-1})}_{\frac{|\mathrm{disc}(K)|^{1/2}}{2^{r_2} \cdot \mathrm{Nm}(\mathfrak{b})}}} \cdot \frac{T}{\mathrm{Nm}(\mathfrak{b})}$$

$\square$