Lemma 4.1.3 Let $\Lambda' \subseteq \Lambda$ be the lattice spanned by a directional basis $(v_1, \dots, v_n)$ of $\Lambda$.

Then, $[\Lambda : \Lambda'] \leq n!$

Pf HW $\square$

**Lemma 4.1.4** There is a basis $(b_1, \dots, b_n)$ of $\Lambda$ with $|b_i| \asymp_n \lambda_i(\Lambda)$.

**Pf** Construct $b_1, \dots, b_n$ iteratively. Assume we've constructed $b_1, \dots, b_{i-1} \in \Lambda$ so that the rank $i-1$ lattice $\Lambda \cap (\mathbb{R} b_1 + \dots + \mathbb{R} b_{i-1})$ is spanned by $b_1, \dots, b_{i-1}$. ~~Let $b_1, \dots, b_i$ be a basis of~~

Let $v \in \Lambda$ be lin. indep. from $b_1, \dots, b_{i-1}$ with $|v| = \lambda_i$.

Let $b_1, \dots, b_i$ be a basis of $\Lambda \cap (\mathbb{R} b_1 + \dots + \mathbb{R} b_{i-1} + \mathbb{R} v)$

Write $b_i = x_1 b_1 + \dots + x_{i-1} b_{i-1} + y v$.

$$\left.\begin{array}{l} \text{W.l.o.g. } 0 \le x_i < 1. \\ v \in \Lambda \Rightarrow \frac{1}{y} \in \mathbb{Z} \Rightarrow |y| \le 1 \end{array}\right\} \Rightarrow |b_i| \le |b_1| + \dots + |b_{i-1}| + |v|$$

$$= \lambda_1 + \dots + \lambda_{i-1} + \lambda_i \le i \cdot \lambda_i \ll_n \lambda_i.$$

□

**Lemma 4.1.5** Let $(v_1, \dots, v_n)$ be a basis of $\Lambda$ with $|b_i| \asymp \lambda_i$.

Let $w = x_1 v_1 + \dots + x_n v_n \quad (x_1, \dots, x_n \in \mathbb{R})$.

Then, $|w| \asymp_{n, |\cdot|} \max(|x_1|\lambda_1, \dots, |x_n|\lambda_n)$

$\asymp_n |x_1|\lambda_1 + \dots + |x_n|\lambda_n$.

**Pf** "$\ll$": triangle inequality.

"$\gg$": Cramer's rule:

$$x_i = \frac{\det\begin{pmatrix} | & & | & | & | & & | \\ v_1 & \cdots & v_{i-1} & w & v_{i+1} & \cdots & v_n \\ | & & | & | & | & & | \end{pmatrix}}{\det\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}}$$

$$\Rightarrow |x_i| \le \frac{|v_1| \cdots |v_{i-1}|\, |w|\, |v_{i+1}| \cdots |v_n|}{\operatorname{covol}(\Lambda)} = \underbrace{\frac{\lambda_1 \cdots \lambda_n}{\operatorname{covol}(\Lambda)}}_{\asymp 1} \cdot \frac{|w|}{\lambda_i}$$

□

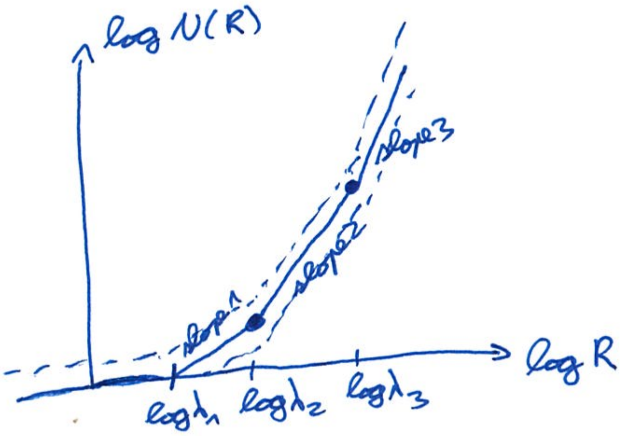
## 4.2. Counting lattice points

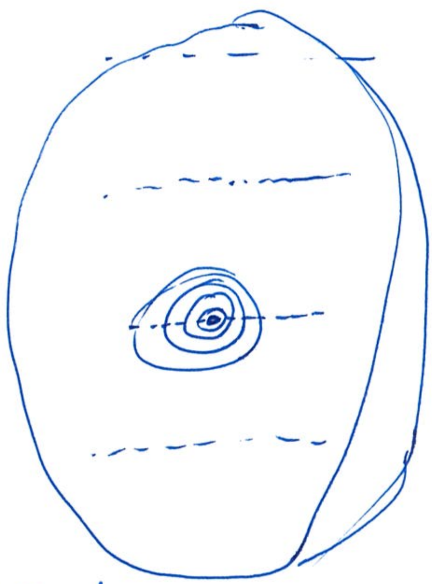
**Thm 4.2.1.**

For any $R \geq 0$:

$$N(R) := \#(\Lambda \cap B(R)) \asymp_{n,|\cdot|} \max_{0\leq i\leq n}\left(\frac{R}{\lambda_1}\cdots\frac{R}{\lambda_i}\right)$$

$$= \begin{cases} 1, & R < \lambda_1 \\ \frac{R}{\lambda_1}\cdots\frac{R}{\lambda_i}, & \lambda_i \leq R < \lambda_{i+1} \\ \frac{R}{\lambda_1}\cdots\frac{R}{\lambda_n}, & \lambda_n \leq R \end{cases}$$



**Pf** Choose a basis $(b_1, \dots, b_n)$ as before and write $w = x_1 b_1 + \dots + x_n b_n$.

$$\left(|w| \leq R \Rightarrow |x_i| \ll \frac{R}{\lambda_i} \ \forall i\right) \Rightarrow \#\{w\} \ll \left(\frac{R}{\lambda_1}+1\right)\cdots\left(\frac{R}{\lambda_n}+1\right)$$

$$\left(|x_i| \ll \frac{R}{\lambda_i} \ \forall i \Rightarrow |w| \leq R\right) \Rightarrow \#\{w\} \gg \left(\frac{R}{\lambda_1}+1\right)\cdots\left(\frac{R}{\lambda_n}+1\right)$$

$$\boxed{\#\{x_i \in \mathbb{Z} \mid |x_i| \leq r\} \asymp 1 + r \text{ for all } r \geq 0}$$

$$\asymp \max_{0\leq i\leq n}\left(\frac{R}{\lambda_1}\cdots\frac{R}{\lambda_i}\right) \qquad \square$$

**Thm 4.2.2** (Davenport's Lemma)

[Reference: Davenport: On a principle of Lipschitz
(+ corrigendum)]

(Let $C \geq 1$.) Let $A \subseteq \mathbb{R}^n$ be a (semialgebraic) set defined by at most $C$ polynomial inequalities $F_i(x_1, \dots, x_n) \geq 0$, each of (total) degree $\leq C$.
For every subset $S = \{i_1, \dots, i_k\}$ of $\{1, \dots, n\}$, let

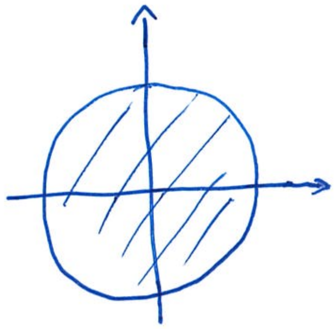
$$\pi_S : \mathbb{R}^n \longrightarrow \mathbb{R}^k, \quad (x_1, \dots, x_n) \longmapsto (x_{i_1}, \dots, x_{i_k})$$

[the proj. that forgets the coordinates not in $S$]

Then,

$$\#(\mathbb{Z}^n \cap A) = \mathrm{vol}_n(A) + O_C\Big(\sum_{\substack{S \subsetneq \{1,\dots,n\} \\ \#S = k}} \mathrm{vol}_k(\pi_S(A))\Big).$$

**Ese** $A = \{(x, y) \mid x^2 + y^2 \leq R^2\}$



$\mathrm{vol}_2(A) = \mathrm{area}(A) = \pi R^2$

$\pi_{\{1\}}(A) = [-R, R] \subseteq \mathbb{R}^1 \rightarrow O(R)$

$\pi_{\{2\}}(A) = [-R, R] \subseteq \mathbb{R}^1 \rightarrow O(R)$

$\pi_{\emptyset}(A) = \{*\} \subseteq \mathbb{R}^0 \rightarrow O(1)$

$$\#(\mathbb{Z}^2 \cap A) = \pi R^2 + O(R + 1)$$

important if $R \to 0$.

Idea of proof, illustrated by this example

Let $I_{x_0} = \{ y \in \mathbb{R} \mid x_0^2 + y^2 \le R^2 \}$

$$\#(\mathbb{Z}^2 \cap A) = \sum_{\substack{x_0 \in \mathbb{Z}: \\ |x_0| \le R}} \#(I_{x_0} \cap \mathbb{Z})$$

$$= \sum_{\substack{x \in \mathbb{Z}: \\ |x| \le R}} \left( \int_{I_x} dy + O(1) \right)$$

$$= \sum_{x} \int_{I_x} dy + O(R+1)$$

$$= \int_{-R}^{R} \int_{I_x} dy\, dx + O(R+1)$$

$$= \pi R^2 + O(R+1).$$

□

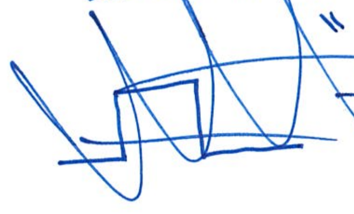
Rmk The general proof uses induction over n and a cell decomposition argument (real algebraic geometry).

Rmk Another interesting point-counting lemma can be found in Widmer: Counting primitive points of bounded height (section 5)

Instead of counting lattice points in a region, one often obtains better error bounds when counting with a smooth weight.

$$\cancel{\#(I\cap \mathbb{Z}) = \text{length}(I) + O(1)}$$

$$\cancel{= \sum_{x\in\mathbb{Z}} 1_I(x)}$$



I.e.: instead of $\#(A\cap \mathbb{Z}^n) = \sum_{x\in\mathbb{Z}^n} 1_A(x)$ $(\approx \mathrm{vol}(A))$

estimate $\sum_{x\in\mathbb{Z}^n} f(x)$ $\left(\approx \int_{\mathbb{R}^n} f(x)\,dx\right)$ for a smooth compactly supported function $f:\mathbb{R}^n\to\mathbb{R}_{\geq 0}$ approximating $1_A$ (from above or below).

$1_I$

To estimate $\sum_{x\in\mathbb{Z}^n} f(x)$, one uses:

<u>Thm 4.2.3</u> (Poisson summation) For any Schwartz function $f:\mathbb{R}^n\to\mathbb{C}$, we have

$$\sum_{x\in\mathbb{Z}^n} f(x) = \sum_{t\in\mathbb{Z}^n} \hat{f}(t).$$

Here, $\hat{f}(0) = \int_{\mathbb{R}^n} f(x)\,dx$ and the remaining terms produce the error term.

**Thm 4.2.4** If $f:\mathbb{R}^n\to\mathbb{C}$ is a Schwartz fct. (e.g. smooth and compactly supported) then $\hat f:\mathbb{R}^n\to\mathbb{C}$ is a Schwartz fct. (in part., $\hat f(t)\ll_{f,k}|t|^{-k}$ $\forall t\in\mathbb{R}^n$, $\forall k\geq 0$).

Let $GL_n(\mathbb{R})$ act on functions $f:\mathbb{R}^n\to\mathbb{C}$ by $(Mf)(x)=f(M^{-1}x)$.

Rmk $\widehat{Mf}=|\det(M)|\cdot(M^T)^{-1}\hat f$.

**Thm 4.2.5** Let $f:\mathbb{R}^n\to\mathbb{C}$ be smooth and compactly supported, and $k\geq 1$. Let $f_R(x)=f\left(\frac{x}{R}\right)$ $(I_R f)(x)$, $I_R = \begin{pmatrix} R & & 0\\ & \ddots & \\ 0 & & R\end{pmatrix}$. Then

$$\sum_{x\in\mathbb{Z}^n} f_R(x) = R^n\cdot \int_{\mathbb{R}^n} f(x)\,dx + O_{f,k}(R^{-k}) \text{ for } R\to\infty.$$

the error goes to 0 !!!

SKIP

Pf $\hat f_R(t)=R^n\cdot \hat f(Rt)$

$$\sum_{0\neq t\in\mathbb{Z}^n}\hat f(Rt)\ll R^{-k}\underbrace{\sum_{0\neq t\in\mathbb{Z}^n}|t|^{-k}}_{\text{indep. of } R,\ <\infty \text{ for large enough } k}\ll R^{-k}$$

$$\Rightarrow \sum f_R(x)=R^n\underbrace{\hat f(0)}_{\int f(x)dx}+O(R^{n-k}).$$

□

More generally, we can count on any lattice:

**Thm 4.2.6** Let $f$ be smooth and cpt supp. and $k \geq 1$. Let $\Lambda$ be a full lattice in $\mathbb{R}^n$ with succ. min. $\lambda_1, \dots, \lambda_n$. Then,

$$\sum_{x \in \Lambda} f(x) = \cancel{\boxed{\frac{\int_{\mathbb{R}^n} f(x)\,dx}{\operatorname{covol}(\Lambda)} + O_{f,k}(\lambda_n^k)}} \quad \text{as } \lambda_n \to 0.$$

$$\frac{1}{\operatorname{covol}(\Lambda)}\left(\int_{\mathbb{R}^n} f(x)\,dx + O_{f,k}(\lambda_n^k)\right)$$

Pf Let $v_1, \dots, v_n$ be a basis of $\Lambda$ with $|v_i| \asymp \lambda_i$.

$$M = \begin{pmatrix} - v_1 - \\ \vdots \\ - v_n - \end{pmatrix}. \quad \Rightarrow \Lambda = M^T \mathbb{Z}^n$$

$$LHS = \sum_{x \in M^T \mathbb{Z}^n} f(x) = \sum_{x \in \mathbb{Z}^n} \underbrace{f(M^T x)}_{((M^T)^{-1} f)(x)} = \sum_{0 \neq t \in \mathbb{Z}^n} \underbrace{|\det(M)|^{-1}}_{\frac{1}{\operatorname{covol}(\Lambda)}} \hat{f}(M^{-1} t)$$

$$\text{Main term} = \frac{1}{\operatorname{covol}(\Lambda)} \int f(x)\,dx.$$

The entries of $M$ are $\ll \lambda_n$.

$$\Rightarrow 1 \leq |t| \ll \lambda_n \cdot |M^{-1} t|$$

$$\Rightarrow \text{error term} \ll \frac{1}{\operatorname{covol}} \sum_{0 \neq t \in \mathbb{Z}^n} \lambda_n^k |t|^{-k} \ll \frac{\lambda_n^k}{\operatorname{covol}}$$

↑ suff. large $k$

$\square$

Rmk

$$\boxed{\text{The previous theorem follows by letting } \Lambda = \tfrac{1}{R}\mathbb{Z}^n.}$$

Rmk A useful way to approximate a fct. $f$ by a smooth fct. is to consider the convolution $f * g$ with a smooth fct. $g: \mathbb{R}^n \to \mathbb{R}$ with (small) cpt. support and with $\int g(x)\,dx = 1$.

$$(f*g)(x) = \int f(x-y)\,g(y)\,dy = \int f(y)\,g(x-y)\,dy$$

Rmk Let $f, g \in L^1(\mathbb{R}^n)$. Then:

a) $f*g \in L^1(\mathbb{R}^n)$

b) $\widehat{f*g}\,(t) = \hat{f}(t) \cdot \hat{g}(t)$

c) $\operatorname{supp}(f*g) \subseteq \operatorname{supp}(f) \cdot \operatorname{supp}(g)$

d) If $g$ is smooth, then $f*g$ is smooth.