Math 229: Introduction to Analytic Number Theory

Spring 2022

Problem set #9

due Friday, April 15 at noon

Problem 1. a) Let $\omega(n)$ be the number of prime divisors of n, counted with multiplicity. Show that for large x, we have

$$\#\{n \leq x : \omega(n) \leq 2\} \asymp \frac{x \log\log x}{\log x}.$$

b) Fix some $0 < \varepsilon < 1$. Show that for large x (depending on ε), we have

$$\#\{n \leq x \text{ not divisible by any prime } p < x^{\varepsilon}\} \asymp_{\varepsilon} \frac{x}{\log x}.$$

Problem 2 (Truncated inclusion–exclusion)**.** a) Let $n \geq 1$ and $k \geq 0$. Show that

$$\sum_{l=0}^{k} (-1)^l \binom{n}{l}$$

is ≥ 0 if k is even and ≤ 0 if k is odd. (We use the convention that $\binom{n}{l} = 0$ if $l > n$.)
Hint: There's a simpler formula for this sum.

b) Use a) to show that if $S_1, \ldots, S_r \subseteq A$ are finite sets, and we let

$$F_k := \sum_{\substack{I \subseteq \{1,\ldots,r\}: \\ \#I \leq k}} (-1)^{\#I} \cdot \#\left(\bigcap_{i \in I} S_i\right),$$

(with the convention that $\bigcap_{i\in\emptyset} S_i = A$), then

$$\#(A \setminus (S_1 \cup \cdots \cup S_r)) \leq F_k \qquad \text{for even } k$$

and

$$\#(A \setminus (S_1 \cup \cdots \cup S_r)) \geq F_k \qquad \text{for odd } k.$$

Problem 3. Show that for sufficiently large $t \geq 1$, for all $k \geq 1$, for sufficiently large x (depending on k), there is a number $x \leq y \leq x + x^{1/k}$ which is the product of at most tk primes.

Problem 4. Show that for sufficiently large $t \geq 1$, for large x, we have

$$\#\{q \leq x \text{ prime} : q + 2 \text{ not divisible by any prime } p < x^{1/t}\} \gg \frac{x}{(\log x)^2}.$$

You may assume the Generalized Riemann Hypothesis.
Remark: This can easily be made unconditional by using the *Bombieri–Vinogradov theorem* instead of the Riemann Hypothesis.