Math 229: Introduction to Analytic Number Theory

Spring 2022

Problem set #8

due Friday, April 8 at noon

Problem 1. Let $f \in \mathbb{Z}[X]$ be a polynomial of degree 1 or 2. Show that

$$\#\{1 \le n \le x : f(n) \text{ squarefree}\} \sim x \cdot \prod_p \frac{\#\{a \in \mathbb{Z}/p^2\mathbb{Z} : p^2 \nmid f(a)\}}{p^2}$$

for $x \to \infty$.
Hints: Use the basic sieve for primes $p \le z$, where z grows slowly with x. Bound the number of $n \le x$ such that $f(n)$ is squarefree but not divisible by p^2 for any $p \le z$ by

$$\sum_{p>z} \#\{n \le x : p^2 \mid f(n)\}.$$

Use that $\#\{a \in \mathbb{Z}/p^2\mathbb{Z} : p^2 \mid f(a)\} \le 2$ for sufficiently large p by Hensel's lemma. (You don't need to show this.)

Problem 2. Let $a_1, a_2, \ldots$ be a multiplicative sequence with $a_p > 1$ for all primes p, let $b_1, b_2, \ldots$ be the sequence with $a = b * \mathbb{1}$, and let $c_1, c_2 \ldots$ be the completely multiplicative sequence given by $c_p = a_p$ for primes p. Show that

$$U(z) := \sum_{\substack{n \le z \\ \text{squarefree}}} \frac{1}{b_n} \ge \sum_{n \le z} \frac{1}{c_n}.$$

Problem 3. a) Let $n \ge 4$. Show that

$$\#\{(p_1, p_2) : p_1, p_2 \text{ prime},\ n = p_1 + p_2\} \ll \frac{n \cdot 2^{\nu(n)}}{(\log n)^2}.$$

Hint: Use the Selberg sieve.

b) (bonus) Improve the upper bound to

$$\ll \frac{n}{(\log n)^2} \cdot \prod_{p|n} \left(1 + \frac{1}{p}\right) \ll \frac{n \log\log n}{(\log n)^2}.$$

Problem 4. a) Let $x \geq k \geq 1$ and $a \geq 1$. Show that

$$\#\{p \leq x,\ p \equiv a \mod k \text{ prime}\} \ll \frac{x}{\varphi(k)\log(2x/k)}.$$

Hint: Use the Selberg sieve.

b) Let $d(n)$ be the number of positive divisors of n. Show that

$$\sum_{p \leq x \text{ prime}} d(p-1) \ll x.$$

Problem 5 (ungraded)**.** Prove Theorem 11.2.2.