Math 229: Introduction to Analytic Number Theory

Spring 2022

Problem set #7

due Friday, April 1 at noon

Problem 1. a) Let $k > 0$ be an integer and let $x, c > 0$. Show that

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{x^s}{s^{k+1}}\mathrm{d}s = \begin{cases} \frac{(\log x)^k}{k!}, & x \geq 1, \\ 0, & x \leq 1. \end{cases}$$

b) (bonus) More generally, for any real numbers $k, x, c > 0$, show that

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{x^s}{s^{k+1}}\mathrm{d}s = \begin{cases} \frac{(\log x)^k}{\Gamma(k+1)}, & x \geq 1, \\ 0, & x \leq 1. \end{cases}$$

(Here, we use a branch of $s^{k+1} = e^{(k+1)\log s}$ which is holomorphic in $\{\Re(s) > 0\}$.)

Problem 2. Show that there is a constant $C > 0$ such that for all $q_1, q_2 \geq 1$ and all primitive real characters χ_1, χ_2, if $q_1 \neq q_2$ or $\chi_1 \neq \chi_2$, then at most one of the functions $L(s, \chi_1)$ and $L(s, \chi_2)$ has a real zero ρ with

$$\Re(\rho) > 1 - \frac{C}{\log(q_1 q_2)}.$$

Hint: Consider the function $\zeta(s)L(s,\chi_1)L(s,\chi_2)L(s,\chi_1\chi_2)$. What is its logarithmic derivative?

Problem 3. As in problem 2a on problem set 3, let $\nu(n)$ be the number of primes dividing n.

a) Show that for any $\varepsilon > 0$, we have $2^{\nu(n)} \ll_\varepsilon n^\varepsilon$.

b) Show that for $\Re(s) > 1 + \varepsilon$, we have $|\zeta(s)| \gg_\varepsilon 1$.

c) Show that there are numbers $A, B \in \mathbb{R}$ with $A \neq 0$ and $\varepsilon > 0$ such that for large x, we have

$$\sum_{n \leq x} 2^{\nu(n)} = Ax \log x + Bx + \mathcal{O}(x^{1-\varepsilon}).$$

Problem 4. Prove the Riemann hypothesis. You may assume that for any $\varepsilon > 0$, for large x, we have

$$\sum_{n \leq x} \Lambda(n) = x + \mathcal{O}_\varepsilon(x^{1/2+\varepsilon}).$$