Math 229: Introduction to Analytic Number Theory

Spring 2022

Problem set #3

due Wednesday, February 16 at noon

Problem 1. a) Show the following, which is slightly stronger than the version of Theorem 3.1.2 stated in class: If $\sum_{n=1}^{N} \frac{a_n}{n^s}$ is bounded (for $N \to \infty$), then $\sum_{n=1}^{\infty} \frac{a_n}{n^{s'}}$ is uniformly convergent in the set

$$\{s' \in \mathbb{C} : \Re(s' - s) \geq \varepsilon \text{ and } |\Im(s' - s)| \leq H \cdot \Re(s' - s)\}$$

for any (fixed) numbers $H > 0$ and $\varepsilon > 0$.

b) Show that a) fails if you replace the assumption that $\sum_{n=1}^{N} \frac{a_n}{n^s}$ is bounded by the weaker assumption that $|\sum_{n=1}^{N} \frac{a_n}{n^s}|$ doesn't go to ∞ as $N \to \infty$.

Problem 2. a) For any $n \geq 1$, let $\nu(n)$ be the number of primes dividing n. Show the following identity of formal Dirichlet series:

$$\sum_{n \geq 1} \frac{2^{\nu(n)}}{n^s} = \zeta(s)^2/\zeta(2s).$$

b) Show the following identity of formal power series:

$$\sum_{n \geq 1} \frac{d(n)^2}{n^s} = \zeta(s)^4/\zeta(2s).$$

c) Let $\Lambda(n) = \log p$ for prime powers $n = p^e$ (with $e \geq 1$) and let $\Lambda(n) = 0$ for all other integers n. Show the following identity of formal Dirichlet series:

$$\sum_{n \geq 1} \frac{\Lambda(n)}{n^s} = -\zeta'(s)/\zeta(s).$$

Problem 3. Let $f(s)$ be the holomorphic continuation of $\zeta(s) - \frac{1}{s-1}$ to the complex plane. For every $s \in \mathbb{C}$ with $\Re(s) > 0$, show that

$$f(s) = \lim_{x\to\infty} \left(\sum_{n\le x} \frac{1}{n^s} - \int_1^x \frac{1}{t^s} \mathrm{d}t \right).$$

Problem 4. Consider the Dirichlet series

$$D(a,s) = \prod_p \left(1 + \frac{42}{p^s} + \frac{123p^{1/2}}{p^{2s}} + \sum_{k\ge 3} \frac{k^{1001}}{p^{ks}} \right).$$

a) What is its abscissa of convergence?

b) Show that it can be meromorphically continued to $\{s \in \mathbb{C} : \Re(s) > \frac{1}{2}\}$.

c) What are the poles of the meromorphic continuation? What order do the poles have?