Math 229: Introduction to Analytic Number Theory

Spring 2022

Problem set #2

due Wednesday, February 9 at noon

Problem 1. Show that the *theta function* $\theta : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ given by

$$\theta(x) = \sum_{n \in \mathbb{Z}} e^{-\pi x n^2}$$

satisfies the functional equation

$$\theta(1/x) = x^{1/2}\theta(x) \qquad \text{for } x > 0.$$

Hint: Use Poisson summation.

Problem 2. Let $c > 0$.

a) Compute the Fourier transform of the function given by $f_c(x) = e^{-c|x|}$.

b) Show that

$$\frac{e^c + 1}{e^c - 1} = \sum_{n=-\infty}^{\infty} \frac{2c}{c^2 + 4\pi^2 n^2}.$$

c) Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Problem 3. Let $\Lambda \subset \mathbb{R}^n$ be a *lattice of rank n*: the $\mathbb{Z}$-module generated by a basis $v_1, \ldots, v_n$ of $\mathbb{R}^n$. Its *covolume* is the volume of the parallelepiped spanned by $v_1, \ldots, v_n$, or equivalently $|\det M|$, where M is the matrix with columns $v_1, \ldots, v_n$. Its *dual lattice* $\widehat{\Lambda}$ is the set of vectors $w \in \mathbb{R}^n$ such that $v_i \cdot w \in \mathbb{Z}$ for all i.

a) Show that $\widehat{\Lambda} \subset \mathbb{R}^n$ is a lattice of rank n.

b) Let $f : \mathbb{R}^n \to \mathbb{C}$ be a Schwartz function. Show that

$$\sum_{x \in \Lambda} f(x) = \frac{1}{\operatorname{covol}(\Lambda)} \sum_{t \in \widehat{\Lambda}} \widehat{f}(t).$$

Further reading: In *New upper bounds on sphere packings*, Cohn and Elkies used the Poisson summation formula to derive upper bounds for the density of sphere-packings. (See Theorem 3.1.) In 2016, Viazovska used this technique to completely solve the sphere-packing problems in dimensions 8 and in dimension 24 (with Cohn, Kumar, Miller, Radchenko).

Problem 4. Show that the Fourier transform of the indicator function of the unit disc $B(1) \subset \mathbb{R}^2$ satisfies

$$\widehat{\mathbb{1}_{B(1)}}(t) = \Omega_{|t| \to \infty}(|t|^{-3/2}).$$

Hint: As in class, let $f(x) = \sqrt{1 - x^2}$ for $-1 \leq x \leq 1$ and $f(x) = 0$ otherwise. Near the singularities, approximate $f'(x)$.