Math 229: Introduction to Analytic Number Theory

Spring 2022

Problem set #10

due Friday, April 29 at noon

Problem 1. a) Which functions $c : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$ that vanish on exactly ω of the residue classes satisfy the following equality?

$$\sum_{t \in (\mathbb{Z}/p\mathbb{Z})^\times} |\widehat{c}(t)|^2 = |\widehat{c}(0)|^2 \cdot \frac{\omega}{p-\omega}.$$

b) Prove Corollary 11.4.3: Let $d \geq 1$ be squarefree and let $c : \mathbb{Z}/d\mathbb{Z} \to \mathbb{C}$ be a function. Assume that for each $p \mid d$, there are $\omega(p)$ residue classes a mod p such that $c(x) = 0$ whenever $x \equiv a \mod p$. Then,

$$\sum_{t \in (\mathbb{Z}/d\mathbb{Z})^\times} |\widehat{c}(t)|^2 \geq |\widehat{c}(0)|^2 \cdot \prod_{p \mid d} \frac{\omega(p)}{p - \omega(p)}.$$

Problem 2 (bonus)**.** Let $f \in \mathcal{S}(\mathbb{R})$ and let $\mathcal{P}$ be a finite set of prime numbers and q be the product of the elements of $\mathcal{P}$. For each $p \in \mathcal{P}$, let $c_p : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$ be a function. Show that

$$\sum_{x \in \mathbb{Z}} f(x) \prod_{p \in \mathcal{P}} c_p(x \bmod p) = \sum_{t \in \mathbb{Z}} \widehat{f}(t/q) \prod_{p \in \mathcal{P}} \widehat{c_p}(t \bmod p).$$

Problem 3. Show that for any integer $k \geq 2$,

$$\int_{\mathbb{R}} \left(\frac{e(t)-1}{2\pi i t} \right)^k e(-t) \mathrm{d}t = \frac{1}{(k-1)!}.$$

Hint: This is the value at 1 of the Fourier transform of the Fourier transform of a reasonably simple function.

Problem 4. Let $N \geq 1$ and let

$$f(t) = \sum_{n=0}^{N} e(tn^2).$$

Let $q \geq 1$ and $a \in (\mathbb{Z}/q\mathbb{Z})^\times$ and $t = a/q$.

a) Assume that q is odd and squarefree. Show that

$$f(t) = \begin{cases} N \cdot \left(\frac{a}{q}\right)/\sqrt{q} + \mathcal{O}(q), & \left(\frac{-1}{q}\right) = 1, \\ N \cdot \left(\frac{a}{q}\right) \cdot i/\sqrt{q} + \mathcal{O}(q), & \left(\frac{-1}{q}\right) = -1. \end{cases}$$

b) Show that

$$f(t)^2 \ll N^2 q^{-1} + N \log q + q \log q.$$

Hint: Expand the product $|f(t)|^2 = f(t)\overline{f(t)}$. Rewrite it as a sum of geometric series.

Problem 5 (bonus)**.** Let $U, V \geq 1$.

a) Show that

$$\mu * \Lambda = \mu * \Lambda_{\leq V} + \mu_{\leq U} * \Lambda - \mu_{\leq U} * \Lambda_{\leq V} + \mu_{>U} * \Lambda_{>V}.$$

b) Prove Vaughan's identity: For any $U, V \geq 1$,

$$\Lambda = \Lambda_{\leq V} + \mu_{\leq U} * L - \mu_{\leq U} * \Lambda_{\leq V} * \mathbb{1} - (\mu_{\leq U} * \mathbb{1})_{>U} * \Lambda_{>V}.$$