Other applications of the circle method:

1) Asymptotics for the nr. of partitions of an integer n with $n \to \infty$.

2) Other weak forms of Goldbach's conjecture, such as $n = p_1 + p_2 + k^2$, ...
(Reference: Vaughan's book)

3) Number of ways of writing $n = a_1 x_1^2 + \dots + a_k x_k^2$ for fixed $0 < a_1, \dots, a_k \in \mathbb{Z}$, varying $x_1, \dots, x_k \in [-N^{1/2}, N^{1/2}]$, for $n \to \infty$.
(Reference: Heath-Brown: A new form of the circle method, and its applications to quadratic forms.)

4) Waring's problem: Every (suff. large) $n \in \mathbb{Z}$ can be written as a sum of at most $c(k)$ k-th powers.

What's the smallest such $c(k)$?

(Reference: Vaughan's book)

⋮

13. Equidistribution

References

- Chapter 11 in Murty
- Noam Elkies's lecture notes

Thm/Def 13.1 A sequence $a_1, a_2, \ldots \in \mathbb{R}/\mathbb{Z}$ is equidistributed / uniformly distributed if the following equivalent statements hold:

a) For all (open/closed/arbitrary) intervals $I \subseteq \mathbb{R}/\mathbb{Z}$,

$$\lim_{N\to\infty} \frac{1}{N} \#\{1 \le n \le N : a_n \in I\} = \operatorname{length}(I).$$

b) For every (piecewise) continuous function $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ (or $\mathbb{C}$),

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(a_n) = \int_{\mathbb{R}/\mathbb{Z}} f(x)\,dx.$$

c) For all $0 \neq t \in \mathbb{Z}$ (or all $0 < t \in \mathbb{Z}$),

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(t a_n) = 0.$$

Rmk There are other notions of equidistribution. E.g.:

A sequence $(S_N)_N$ of (multi-)sets $S_N \subseteq \mathbb{R}/\mathbb{Z}$ is equidistr. if a) $\forall I$:

$$\lim_{N\to\infty} \frac{1}{|S_N|} \sum_{\substack{x \in S_N: \\ x \in I}} (\text{mult. of } x \text{ in } S_N) = \text{length}(I)$$

b) ...

c) ...

A sequence $(\mu_N)_N$ of measures on $\mathbb{R}/\mathbb{Z}$ is equidist. if (weakly conv. to Lebesgue measure)

a) $\forall I$: $\lim_{N\to\infty} \int_{\mathbb{R}/\mathbb{Z}} d\mu_N = \text{length}(I)$

b) ...

c) ...

Or a sequence could be equidistr. w.r.t. some other (non-Lebesgue) measure on $\mathbb{R}/\mathbb{Z}$.

Pf (sketch) b) $\Rightarrow$ c): clear

b) $\Rightarrow$ a): Approximate 1_I by continuous functions.

a) $\Rightarrow$ b): Approximate f by step functions
(= linear combinations of indicator functions 1_I).

c) $\Rightarrow$ b): Approximate f by functions of the form $\sum_{t=-M}^{M} b_t \, e(tx)$.

$\square$

Rmk c) looks like a qualitative version of the kind of estimate we needed for minor arcs in the circle method.

b) is also useful. For example, we previously wrote $\sum_{n \le N} d(n) = \sum_{a \le N} \left\lfloor \frac{N}{a} \right\rfloor = \sum_{a \le N} \frac{N}{a} - \sum_{a \le N} \left\{ \frac{N}{a} \right\}$.

~~If the fractional parts $\left\{\frac{N}{a}\right\}$ were equidistributed~~

Let $S_N = \left\{ \left\{ \frac{N}{a} \right\} : 1 \le a \le N \right\}$.
If the sequence $(S_N)_N$ were equidistributed, then
$\sum_{a \le N} \left\{ \frac{N}{a} \right\} \sim N \cdot \int_0^1 x \, dx = \frac{1}{2} N$.
(But we've previously seen that this is false!)

Ex Let $\lambda \in \mathbb{R}$. Then, $a_n = \{\lambda n\}$ is equidistr. if and only if $\lambda \notin \mathbb{Q}$.

Pf "$\Rightarrow$" via a): If $\lambda \in \mathbb{Q}$, then the sequence only takes finitely many values. $\Rightarrow$ There are gaps.
$\Rightarrow$ not equidistributed.

"$\Rightarrow$" via c): If $t\lambda \in \mathbb{Z}$, then

$$\frac{1}{N}\sum_{n=1}^{N} \underbrace{e(t\lambda n)}_{1} = 1 \;\overset{N\to\infty}{\not\longrightarrow}\; 0.$$

"$\Leftarrow$" via c): Since $t\lambda \notin \mathbb{Z}$ and therefore $e(t\lambda) \neq 1$, we have

$$\frac{1}{N}\sum_{n=1}^{N} e(t\lambda n) = \frac{1}{N}\underbrace{e(t\lambda)\cdot\frac{e(t\lambda N)-1}{e(t\lambda)-1}}_{\ll 1} \ll \frac{1}{N} \longrightarrow 0$$

$\square$

Ex The sequence of $S_N = \left\{ \frac{b}{N} \mid b \in \mathbb{Z}/N\mathbb{Z} \right\} \subseteq \mathbb{R}/\mathbb{Z}$

is equidistributed:

$$\frac{1}{N} \sum_{b} e\left(t \frac{b}{N}\right) = 0 \qquad \text{unless } N \mid t.$$

Ex The sequence of $S_N = \left\{ \frac{a}{N} \mid a \in (\mathbb{Z}/N\mathbb{Z})^\times \right\} \subseteq \mathbb{R}/\mathbb{Z}$

is equidistributed:

$$\frac{1}{N} \sum_{b} e\left(t \frac{b}{N}\right) \ll \frac{|t|}{N} \qquad \text{(cf. pf. of Thm 12.2.6).}$$

Ex Let $a_1, a_2, \ldots$ be the (reduced) fractions $\frac{b}{q} \in [0,1)$ sorted by $q \geq 1$

and in case of ties by b:

$$\frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \ldots$$

This sequence is equidistributed.

Thm 13.2 (van der Corput; Weyl differencing trick)

If the sequence $(a_{n+d}-a_n)_n$ is equidistributed for all $d\geq 1$, then $(a_n)_n$ is equidistributed.

First attempt of a pf

$$\left|\sum_{n=1}^{N} e(ta_n)\right|^2 = \sum_{n=1}^{N} e(ta_n) \sum_{m=1}^{N} \overline{e(ta_m)}$$

$$= \sum_{n,m} \underbrace{e(t(a_n-a_m))}_{1 \text{ if } n=m}$$

$$= N + \sum_{n\neq m} e(t(a_n-a_m))$$

$$\underset{\substack{\uparrow \\ d=n-m}}{=} N + \sum_{\substack{-N<d<N \\ d\neq 0}} \underbrace{\sum_{\substack{n: \\ 1\leq n\leq N, \\ 1\leq n+d\leq N}} e(t(a_{n+d}-a_n))}_{\substack{\text{looks like the sum in} \\ \text{the def. of equidistr. of } (a_{n+d}-a_n)_n}}$$

Problem: We don't know how quickly

$$\frac{1}{N}\sum_{\substack{n: \\ \dots}} e(t(a_{n+d}-a_n)) \text{ goes to } 0 \text{ for } N\to\infty \text{ as } d \text{ varies.}$$

Solution: Only allow bounded differences.

We'll show the following slightly more general lemma:

<u>Lemma 13.3</u> Let $x_1, \dots, x_N \in \mathbb{C}$, $H \geq 1$. Then,

$$\left|\sum_{n=1}^{N} x_n\right|^2 \leq \frac{H+N}{H+1}\left(\sum_{n=1}^{N} |x_n|^2 + 2\sum_{d=1}^{H}\left(1-\frac{d}{H+1}\right)\left|\sum_{n=1}^{N-d} x_{n+d}\overline{x_n}\right|\right)$$

<u>Pf</u> Let $x_n = 0$ unless $1 \leq n \leq N$.

$$(H+1)^2\left|\sum_n x_n\right|^2 = \left|\sum_{h=0}^{H}\sum_n x_{n+h}\right|^2 = \left|\sum_{n=-H+1}^{N}\sum_{h=0}^{H} x_{n+h}\right|^2$$

$$\leq (H+N)\cdot\sum_n\left|\sum_{h=0}^{H} x_{n+h}\right|^2$$

Cauchy-Schwarz or AM-QM

Here, $$\sum_n\left|\sum_{h=0}^{H} x_{n+h}\right|^2 = \sum_n \sum_{0\leq h,k\leq H} x_{n+h}\overline{x_{n+k}}$$

$$= \sum_n\left\{\sum_h |x_{n+h}|^2 + \sum_n \sum_{\substack{-H\leq d\leq H\\ d\neq 0}} \sum_{\substack{0\leq k\leq H:\\ 0\leq k+d\leq H}} x_{n+k+d}\overline{x_{n+k}}\right\}$$

$$= (H+1)\sum_n |x_n|^2 + 2\,\mathrm{Re}\left(\sum_{d=1}^{H} (H+1-d)\cdot \sum_{n=1}^{N-d} x_{n+d}\overline{x_n}\right)$$

$$\leq (H+1)\sum_n |x_n|^2 + 2\sum_{d=1}^{H}(H+1-d)\cdot\left|\sum_{n=1}^{N-d} x_{n+d}\overline{x_n}\right|.$$

$\square$

Pf of Thm

Use the lemma with $x_n = e(ta_n)$.

$$\Rightarrow \left| \frac{1}{N} \sum_{n=1}^{N} e(ta_n) \right|^2$$

$$\leq \frac{\frac{H}{N}+1}{H+1} \left(1 + 2 \cdot \sum_{d=1}^{H} \left(1 - \frac{d}{H+1}\right) \underbrace{\left| \frac{1}{N} \sum_{n=1}^{N-d} e(t(a_{n+d}-a_n)) \right|} \right)$$

$$\frac{\frac{H}{N}+1}{H+1} \xrightarrow{N\to\infty} \frac{1}{H+1}$$

$$\left| \frac{1}{N} \sum_{n=1}^{N-d} e(t(a_{n+d}-a_n)) \right| \xrightarrow{N\to\infty} 0$$ because $(a_{n+d}-a_n)_n$ is equidistributed

$$\Rightarrow \limsup_{N\to\infty} (\text{LHS}) \leq \frac{1}{H+1} \quad \text{for all } H \geq 1.$$

$$\Rightarrow \lim_{N\to\infty} (\text{LHS}) = 0$$

$\Rightarrow (a_n)_n$ is equidistributed. $\square$

Cor 13.4 Let $f(X) = b_m X^m + \dots + b_0 \in \mathbb{R}[X]$.

Then, $a_n = \{f(n)\}$ is equidistributed if and only if $b_i \notin \mathbb{Q}$ for some $i \geq 1$.

Pf "$\Rightarrow$" If $b_i \in \mathbb{Q}$ for all $i \geq 1$, then $\{f(n)\}$ only takes finitely many values.

"$\Leftarrow$"

We prove the statement by induction over m.

w.l.o.g. $b_0 = 0$.

If $b_m \in \mathbb{Q}$, say $b_m = \frac{p}{q}$,

let $g(x) = f(x) - b_m X$.

$\Rightarrow \deg(g) < m$ and g still has an irrational nonconst. coeff.

$$\sum_{n=1}^{N} e(t f(n)) = \sum_{n=1}^{N} e(t \underbrace{b_m}_{\frac{p}{q}} n) e(t g(n))$$

$$= \sum_{r \in \mathbb{Z}/q\mathbb{Z}} e\left(\frac{tpr}{q}\right) \underbrace{\sum_{\substack{n=1: \\ n \equiv r \bmod q}}^{N} e(t g(n))}$$

$$= \sum_{0 \leq m \leq \left[\frac{N-r}{q}\right]} e(t g(r+qm)).$$

$n = r + qm$
(where $1 \leq r \leq q$)

The pol. $g(r+qX)$ has an irrational nonconst. coeff., so by induction $\frac{1}{N}\sum\limits_{m} e(\cdots) \xrightarrow{N\to\infty} 0$.

$$\Rightarrow \frac{1}{N}\sum_{n=1}^{N} e(t f(n)) \xrightarrow{N\to\infty} 0.$$

<u>If $b_m \notin \mathbb{Q}$:</u> If $m=1$, this is the example $a_n = \{b_1 n\}$, so assume $m \geq 1$.

For any $d \geq 1$, the polynomial $f(x+d)-f(x)$ $= b_m((x+d)^m - x^m) + b_{m-1}((x+d)^{m-1} - x^{m-1}) + \ldots$ of degree $m-1$ has the irrational leading coefficient $m d a_m$.

$\Rightarrow$ By induction, the sequence $(f(n+d)-f(n))_n$ is equidistr. for all $d \geq 1$.

$\Rightarrow$ By the Thm, the sequence $(f(n))_n$ is equidistr. $\square$

<u>Rmk</u> One can, and e.g. when applying the circle method to Waring's problem wants to, estimate the rate of convergence ("the speed of equidistribution").