Lemma 12.2.5 Let $t \in \mathbb{R}$, $X \geq 1$. Then, there is a rat. nr. $\frac{a}{q}$ (with gcd$(a,q)=1$) with $1 \leq q \leq X$ and $|t - \frac{a}{q}| \leq \frac{1}{qX}$.

Pf $|t - \frac{a}{q}| \leq \frac{1}{qX} \Leftrightarrow |qt - a| \leq \frac{1}{X}$.

~~$\Rightarrow$ We want to show that~~ one of the numbers ~~$\|qt\|$ with $1 \leq q \leq X$ has distance ≤ 1 from an integer.~~

$\Rightarrow$ We want to show that $\|qt\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{1}{X}$ for some

$1 \leq q \leq X$.

But clearly, two of the X elements $\{qt\}$ of $\mathbb{R}/\mathbb{Z}$ have distance $\leq \frac{1}{X}$ in $\mathbb{R}/\mathbb{Z}$. Take their difference. $\square$

We can now prove Thm 12.2.1. More precisely:

<u>Thm 12.2.6</u> (Assume the GRH.) For $n \to \infty$,

$$\sum_{\substack{p_1, p_2, p_3: \\ n = p_1 + p_2 + p_3}} \log(p_1)\log(p_2)\log(p_3)$$

$$= \frac{1}{2} n^2 \cdot \prod_{p \nmid n} \left(1 + \frac{1}{(p-1)^3}\right) \cdot \prod_{p \mid n} \left(1 - \frac{1}{(p-1)^2}\right) + o(n^2).$$

<u>Pf of Thm 12.2.1</u> (weak Goldbach)

RHS > 0 for all <u>odd</u> n.

$\Rightarrow$ LHS > 0 for all suff. large odd n. $\square$

Pf of Thm 12.2.6

$$\text{LHS} = \int_{\mathbb{R}/\mathbb{Z}} \hat{f}(t)^3 e(-nt)\,dt.$$

Let $Q = n^{\alpha}$ ~~(in fact, $Q = n^{\alpha}$ for any~~
for some ~~fixed $0 < \alpha < \frac{1}{2}$ should work).~~
sufficiently small $\alpha > 0$.

We let

$$\mathcal{M} = \Big\{ t \in \mathbb{R}/\mathbb{Z} : \text{for some } 1 \le q \le Q,\ a \in (\mathbb{Z}/q\mathbb{Z})^{\times},$$
$$\Big\| t - \frac{a}{q} \Big\|_{\mathbb{R}/\mathbb{Z}} < \frac{1}{2Q^2} \Big\}.$$

(points on major arcs).

The difference between any rat. nrs. with denominator $\le Q$ is $\ge \frac{1}{Q^2}$, so for any $t \in \mathcal{M}$, there is exactly one fraction $\frac{a}{q}$ as above. ("The major arcs are disjoint.")

By Cor 12.2.4.,

$$\int_{\mathcal{M}} \hat{f}(t)^3 e(-nt)\,dt = \sum_{q=1}^{Q} \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^{\times}} \frac{\mu(q)}{\varphi(q)^3} \cdot \frac{n^2}{2} \cdot e\Big(-\frac{an}{q}\Big)$$
$$+ \mathcal{O}(n^{2-\varepsilon})$$

(for some $\varepsilon > 0$).

$$= \sum_{q=1}^{\infty} (\cdots) + \mathcal{O}(n^{2-\varepsilon})$$

Here, $c_q^{(n)} = \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} e\left(-\frac{an}{q}\right) = \sum_{d|q} \mu(d) \sum_{\substack{a \in \mathbb{Z}/q\mathbb{Z}: \\ d|a}} e\left(-\frac{an}{q}\right)$

$$= \sum_{d|q} \mu(d) \underbrace{\sum_{b \in \mathbb{Z}/\frac{q}{d}\mathbb{Z}} e\left(-\frac{bn}{q/d}\right)}_{\substack{\frac{q}{d} \text{ if } \frac{q}{d} | n \\ 0 \text{ otherwise}}}$$

$$\underset{e=\frac{q}{d}}{\overset{=}{\uparrow}} \sum_{\substack{e|q \\ e|n}} e\,\mu\left(\frac{q}{e}\right)$$

is multiplicative in q. If n is divisible by p exactly r times, then

$$\left(\boxed{c_{p^s}^{(n)} = \sum_{0 \le i \le \min(r,s)} p^i \underbrace{\mu(p^{s-i})}_{\substack{1 \text{ if } i=s \\ -1 \text{ if } i=s-1 \\ 0 \text{ if } i \le s-2}} = \begin{cases} p^s - p^{s-1}, & s \le r \\ -p^{s-1}, & s = r+1 \\ 0, & s \ge r+2 \end{cases}}\right)$$

$$c_p(n) = \begin{cases} -1+p, & p|n \\ -1, & p \nmid n \end{cases}$$

$$\Rightarrow \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi(q)^3} \cdot \frac{n^2}{2} \cdot c_q^{(n)} = \frac{n^2}{2} \cdot \prod_p \left(1 - \frac{c_p(n)}{\varphi(p)^3}\right)$$

$$= \frac{n^2}{2} \cdot \prod_{p \nmid n} \left(1 + \frac{1}{(p-1)^3}\right) \cdot \prod_{p|n} \left(1 - \frac{1}{(p-1)^2}\right)$$

Now, we deal with the minor arcs.

By Lemma 12.2.5, for every $t \in \mathbb{R}/\mathbb{Z}$, there is some $\frac{a}{q}$ with $1 \le q \le n^{1-\alpha}$ and $\|t - \frac{a}{q}\| \le \frac{1}{q \cdot n^{1-\alpha}}$.

If $q \le n^{\alpha} = Q$, then $t \in \mathfrak{M}$. Otherwise, $\|t - \frac{a}{q}\| \le \frac{1}{n}$.

This gives an upper bound

$$\hat{f}(t) \ll n^{1-\varepsilon} \quad \text{for some } \varepsilon > 0.$$

This would immediately imply

$$\int_{(\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{M}} (\dots) \ll n^{3-\varepsilon} \quad \text{for some } \varepsilon > 0,$$

which is larger than the main term.

(for small ε)

Solution: We also know that

$$\int |\hat{f}(t)|^2 \, dt = \sum_{x \in \mathbb{Z}} |f(x)|^2 = \sum_{p \le n} \log p \asymp n.$$

Hence,

$$\int_{(\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{M}} (\dots) \ll n \cdot n^{1-\varepsilon} = n^{2-\varepsilon} \quad \text{for some } \varepsilon > 0,$$

which grows more slowly than the main term! $\square$

Remark To get rid of the GRH assumption, use the known zero-free region to obtain an estimate on the major arcs.

Unfortunately, it is only useful for $q \ll e^{c\sqrt{\log n}}$, so we take $Q = e^{c\sqrt{\log n}}$.

For the minor arcs, you need a different way of obtaining an upper bound.

This can be done using the following identity by Vaughan, which may remind you of sieve theory:

Vaughan's identity

For a sequence $a = (a_1, a_2, \dots)$ and any $T \geq 1$, let $a_{\leq T}$, $a_{>T}$ be the sequences with

$$(a_{\leq T})_n = \begin{cases} a_n, & n \leq T \\ 0, & n > T \end{cases} \qquad (a_{>T})_n = \begin{cases} 0, & n \leq T \\ a_n, & n > T. \end{cases}$$

(Clearly, $a = a_{\leq T} + a_{>T}$.)

Then,

$$\Lambda = \Lambda_{\leq V} + \mu_{\leq U} * L - \mu_{\leq U} * \Lambda_{\leq V} * 1 - (\mu_{\leq U} * 1)_{>U} * \Lambda_{>V},$$

where $L_n = \log(n)$.

(See for example Ch. 24 in Davenport or Ch. 3 in Vaughan.)

Thm 12.2.7

Let $f(n) = n \cdot \prod_{p \nmid n} \left(1 - \frac{1}{(p-1)^2}\right) \cdot \prod_{p \mid n} \left(1 + \frac{1}{p-1}\right)$.

Then,

$$\sum_{n=1}^{N} \Big(\sum_{\substack{p_1, p_2 : \\ n = p_1 + p_2}} \log(p_1) \log(p_2) - f(n) \Big)^2 \ll N^{3-\varepsilon}$$

for some $\varepsilon > 0$.

Note This saves a power of N^{ε} compared to the trivial estimate.

Cor 12.2.8

$$\#\{1 \leq n \leq N \text{ even not the sum of two primes}\} \ll N^{1-\varepsilon}$$

for some $\varepsilon > 0$.

Pf of Cor

If n is even and not the sum of two primes, then the summand

$$\left(\sum \cdots - f(n) \right)^2 = f(n)^2 \text{ is } \gg n^2.$$

$\square$

Pf of Thm (sketch) Take $f(k) = \begin{cases} \log(k), & k \le N \text{ prime} \\ 0, & \text{otherwise} \end{cases}$

The major arcs work like before.

For the minor arcs:

$$\sum_{n=1}^{N} \left| \int_{(\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{M}} \hat{f}(t)^2 e(-nt) \right|^2$$

$$\le \sum_{n \in \mathbb{Z}} |\cdots|^2 = \int_{(\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{M}} |\hat{f}(t)|^4 \, dt$$

Fourier transform preserves inner product

This fourth power can be bounded like before.

$\square$