Rmk $F(a, e(t)) = \sum a_n e(nt)$ is the Fourier transform

$$\hat{f} : \mathbb{R}/\mathbb{Z} \to \mathbb{C} \quad \text{of} \quad f : \mathbb{Z} \to \mathbb{C}$$
$$n \mapsto \begin{cases} a_n, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

The prev. remark just describes the inverse Fourier transform.

12.2. Goldbach Conjecture

Conj Every even $n \geq 4$ is the sum of two primes.

Thm (Helfgott; Weak Goldbach Conj)
Every odd $n \geq 7$ is the sum of three primes.

The proof is a book...

We'll only prove:

Thm 12.2.1 (Hardy, Littlewood)
Assume the GRH. Then, every suff. large odd n is the sum of three primes.

Rmk Before Helfgott, Vinogradov removed the GRH assumption.

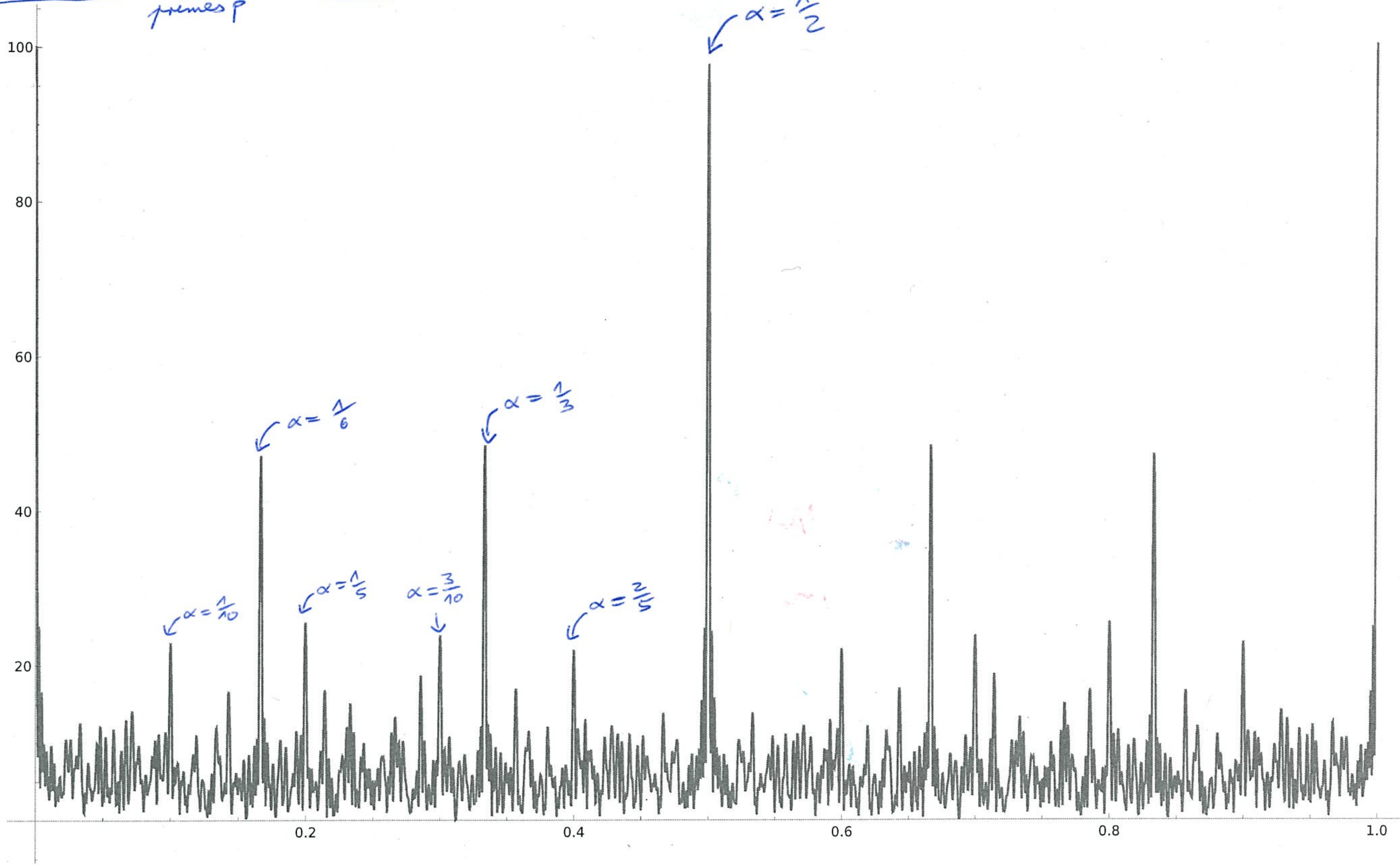
References: - Chapter 26 in Davenport: Mult. NT
- Chapter 3 in Vaughan: The Hardy-Littlewood Circle Method.

Goal: Let
$$f(k) = \begin{cases} 1, & k \overset{n}{\leq} \text{ prime}, \\ 0, & \text{otherwise.} \end{cases}$$

$$\Rightarrow (f * f * f)(n) = \#\{(p_1, p_2, p_3) \text{ prime} : n = p_1 + p_2 + p_3\}.$$
$$\| $$
$$\widehat{f^3}(-n) = \int_{\mathbb{R}/\mathbb{Z}} \hat{f}(t)^3 e(-nt)\, dt$$

Estimate this.

Graph of $\left| \sum_{\substack{\text{first 100} \\ \text{primes } p}} e(pt) \right|$



<u>Observation</u> "$|\hat{f}(t)|$ is largest when t is close to a rational number with small (sqfree) denominator"

$\rightsquigarrow$ The integral $\displaystyle\int_{\mathbb{R}/\mathbb{Z}} \hat{f}(t)^r e(-nt)\,dt$ is (hopefully) dominated by the integral over $t \in \mathbb{R}/\mathbb{Z}$ close to these rat. numbers, at least for $r \geq 3$."

<u>Terminology</u> Major arcs:

Set of pts $t \in \mathbb{R}/\mathbb{Z}$ close to such a rat. nr.

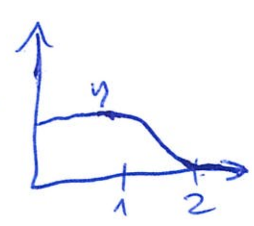
Minor arcs:

Set of other pts.

For simplicity, we'll instead work with the function

$$f(k) = \begin{cases} \log k, & k \text{ is a prime}, \\ 0, & \text{otherwise}. \end{cases}$$

<u>Rmk</u> It's generally also worth considering a <u>smooth cutoff</u>: For example, fix η as in the picture and let



$$f(k) = \begin{cases} (\log k)\,\eta\left(\frac{k}{n}\right), & k \text{ prime}, \\ 0, & \text{otherwise}. \end{cases}$$

Heuristic

$$\hat{f}\left(\frac{a}{q}\right) = \sum_{p \le n} \log(p)\, e\left(\frac{ap}{q}\right)$$

$$= \sum_{r \in (\mathbb{Z}/q\mathbb{Z})^\times} \sum_{\substack{p \le n \\ p \equiv r \bmod q}} \log(p)\, e\left(\frac{ap}{q}\right)$$

$$\approx \sum_{r} \frac{n}{\varphi(q)}\, e\left(\frac{\cancel{a}r}{q}\right)$$

$$= \frac{n}{\varphi(q)} \sum_{d \mid q} \mu(d) \underbrace{\sum_{\substack{r \in \mathbb{Z}/q\mathbb{Z}: \\ d \mid r}} e\left(\frac{r}{q}\right)}_{\substack{1 \text{ if } d = q \\ 0 \text{ if } d \neq q}}$$

$$= \frac{\mu(q)}{\varphi(q)} \cdot n.$$

<u>Lemma 12.2.2</u> Assume the GRH.

Let $q \geq 1$, $a \in (\mathbb{Z}/q\mathbb{Z})^\times$. Then,

$$\hat{f}\left(\frac{a}{q}\right) = \frac{\mu(q)}{\varphi(q)} \cdot n + \mathcal{O}\left(q^{1/2} n^{1/2} (\log n)^2\right).$$

Rmk This is useless for $q \geq n$ because obviously $\hat{f}(t) \ll n$ for all t.

Pf $\hat{f}\left(\frac{a}{q}\right) = \sum_{p \leq n} \log(p)\, e\left(\frac{ap}{q}\right) = \sum_{\substack{k \leq n \\ \gcd(k,q)=1}} \Lambda(k)\, e\left(\frac{ak}{q}\right) + \mathcal{O}\left(q^{1/2} n^{1/2} (\log n)^2\right).$

We can write the function

$$c : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$$
$$t \mapsto \begin{cases} e\left(\frac{t}{q}\right), & t \in (\mathbb{Z}/q\mathbb{Z})^\times, \\ 0, & \text{otherwise} \end{cases}$$

as a lin. comb. of the multiplicative characters $\chi \bmod q$:

$$\frac{1}{\varphi(q)} \sum_{\chi} \overline{\tau(\chi)}\, \chi(-t)$$

$$= \frac{1}{\varphi(q)} \sum_{\chi} \sum_{x} \overline{\chi(x)}\, e\left(-\frac{x}{q}\right) \chi(-t)$$

$$= \frac{1}{\varphi(q)} \sum_{x \in (\mathbb{Z}/q\mathbb{Z})^\times} \underbrace{\sum_{\chi} \chi\left(-\frac{t}{x} \bmod q\right)}_{\substack{\varphi(q) \text{ if } -\frac{t}{x} \equiv 1 \bmod q \\ 0 \text{ otherwise}}} e\left(-\frac{x}{q}\right)$$

$$= \begin{cases} e\left(\frac{t}{q}\right), & t \in (\mathbb{Z}/q\mathbb{Z})^\times \\ 0, & \text{otherwise} \end{cases}$$

$$= c(t).$$

$$\Rightarrow \hat{f}\left(\frac{a}{q}\right) = \sum_{k \le n} \Lambda(k) \, \frac{1}{\varphi(q)} \sum_{\chi} \overline{\tau(\chi)} \, \chi(-ak) + O(\ldots)$$

$$= \frac{1}{\varphi(q)} \sum_{\chi} \overline{\tau(\chi)} \underbrace{\,}_{\cdot \chi(-a) \cdot} \sum_{k \le n} \Lambda(k) \chi(k) \quad + O(\ldots)$$

The GRH implies that

$$\sum_{k \le n} \Lambda(k) \chi(k) = \begin{cases} n, & \chi = \chi_0 \\ 0, & \chi \ne \chi_0 \end{cases} + O(n^{1/2} (\log n)^2).$$

Furthermore, we've already seen in the heuristic that

$$\tau(\chi_0) = \sum_{x \in (\mathbb{Z}/q\mathbb{Z})^\times} e\left(\frac{x}{q}\right) = \mu(q).$$

Also, one can show that $|\tau(\chi)| \le q^{1/2}$ for all characters χ (not necessarily primitive). The claim follows immediately (noting that there are exactly $\varphi(q)$ char. χ). $\square$

Cor 12.2.3 Assume the GRH.

Let $q \geq 1$, $a \in (\mathbb{Z}/q\mathbb{Z})^\times$, $0 \neq s \in \mathbb{R}$

Then,

$$\hat{f}\left(\tfrac{a}{q}+s\right) = \frac{\mu(q)}{\varphi(q)} \cdot \underbrace{\frac{e(sn)-1}{2\pi i s}}_{\left(\xrightarrow{s\to 0} n\right)} + O\left(q^{1/2} n^{1/2} (\log n)^2 (1+sn)\right).$$

Pf We "know" $\hat{f}\left(\frac{a}{q}\right) = \sum_{p \leq n} \log(p)\, e\left(\frac{a}{q}\cdot p\right)$ for all n.

We want $\hat{f}\left(\frac{a}{q}+s\right) = \sum_{p \leq n} \log(p)\, e\left(\frac{a}{q}\cdot p\right) \cdot e(s\cdot p)$

$\rightsquigarrow$ Use Abel summation on the functions

$$g(x) := \sum_{p \leq x} \log(p)\, e\left(\frac{a}{q}\cdot p\right) - \frac{\mu(q)}{\varphi(q)}\cdot x$$

and

$$h(x) := e(sx).$$

$\square$

(Rmk If sn is large, ~~integrating~~ differentiating $e(sx)$ in Abel summation might be ill-advised because it oscillates.)

~~Using the fact that $g(x+d)$ $g(x)$~~

Cor 12.2.4 Assume the GRH. Let $q \geq 1$, $a \in (\mathbb{Z}/q\mathbb{Z})^\times$, $\delta > 0$.

Then,

$$\int_{\frac{a}{q}-\delta}^{\frac{a}{q}+\delta} \hat{f}(t)^3 \, e(-tn)\, dt$$

$$= \frac{\mu(q)}{\varphi(q)^3} \cdot \left(\frac{n^2}{2} e\left(-\frac{an}{q}\right) + \mathcal{O}\left(\frac{1}{(\delta n)^2}\right)\right) + \mathcal{O}\left(\delta \, \frac{q^{1/2} n^{1/2} (\log n)^2 (1+\delta n)}{\varphi(q)^2} + \delta \cdot q^{3/2} n^{3/2} (\log n)^6 (1+\delta n)^3 \right)$$

Pf Just integrate... $\square$