Reminder

Let $n \in \mathbb{Z}$ be a prod. of primes $< z$.

$$\sum_{d \mid n} \mu(d) = \begin{cases} 1, & n = 1 \\ 0, & n \neq 1 \end{cases} \qquad (I)$$

This is an exact sieve for primes $< z$.

If $\lambda_1, \lambda_2, \ldots \in \mathbb{R}$ satisfy for all $n \in \mathbb{Z}$ prod. of pr.

$$\sum_{d \mid n} \lambda_d \geq \begin{cases} 1, & n = 1, \\ 0, & n \neq 1, \end{cases}$$

we get an upper bound sieve. The numbers $(\lambda_n)_n$ are called <u>upper bound sieve coefficients</u>.

If $\ldots \leq \ldots$, we get a lower bound sieve and the numbers $(\lambda_n)_n$ are called <u>lower bound sieve coefficients</u>.

<u>Rmk</u> (I) follows by expanding the product

$$\prod_{p < z} (1 - a_p) = \begin{cases} 1, & n = 1, \\ 0, & n \neq 1, \end{cases}$$

where $a_p = \begin{cases} 1, & p \mid n, \\ 0, & p \nmid n. \end{cases}$

You can "partially expand" a product $\prod_{i=1}^{n}(1+b_i)$

as follows:

Lemma 1.1.3.1 Let $b_1, \dots, b_n \in \mathbb{R}$.

Let $\mathcal{S}$ be a set of subsets of $\{1, \dots, n\}$ s.th.:

a) $\emptyset \in \mathcal{S}$

b) If $\emptyset \neq A \in \mathcal{S}$, then $A \setminus \{\min(A)\} \in \mathcal{S}$.

Then,

$$\prod_{i=1}^{n}(1+b_i) = \sum_{A \in \mathcal{S}} \prod_{i \in A} b_i + \sum_{\substack{A \subseteq \{1, \dots, n\} \\ A \notin \mathcal{S} \\ A \setminus \{\min(A)\} \in \mathcal{S}}} \left(\prod_{i \in A} b_i\right) \cdot \prod_{i=1}^{\min(A)-1}(1+b_i).$$

Pf. Use induction over n, considering the sets

$$\mathcal{T} := \{B \subseteq \{1, \dots, n-1\} : B \in \mathcal{S}\},$$

$$\mathcal{U} := \{B \subseteq \{1, \dots, n-1\} : B \cup \{n\} \in \mathcal{S}\}.$$

...

$\square$

Pf 2 (Jonas)

$$LHS = \prod_{i=1}^{n} (1+b_i) = \sum_{C \subseteq \{1,\dots,n\}} \prod_{i \in C} b_i .$$

$$RHS = \sum_{A \in \mathcal{S}} \prod_{i \in A} b_i + \sum_{\substack{A \subseteq \{1,\dots,n\} \\ A \notin \mathcal{S} \\ A \setminus \{\min(A)\} \in \mathcal{S}}} \left(\prod_{i \in A} b_i \right) \cdot \prod_{i=1}^{\min(A)-1} (1+b_i)$$

$$= \sum_{A \in \mathcal{S}} \prod_{i \in A} b_i + \sum_{A : \dots} \sum_{B \subseteq \{1,\dots,\min(A)-1\}} \prod_{i \in A \cup B} b_i \qquad (I)$$

Consider any subset $C = \{j_1, \dots, j_m\}$ of $\{1,\dots,n\}$, with $j_1 < \dots < j_m$.

By a) and b), there is some $0 \le l \le m$ such that

$\{j_k, \dots, j_m\} \in \mathcal{S}$ if $k > l$,

$\{j_k, \dots, j_m\} \notin \mathcal{S}$ if $k \le l$.

If $l = 0$, then $C \in \mathcal{S}$.

If $l > 0$, then C can be written uniquely as $C = A \cup B$ with $A \notin \mathcal{S}$, $A \setminus \{\min(A)\} \in \mathcal{S}$, $B \subseteq \{1,\dots,\min(A)-1\}$, namely $A = \{j_l, \dots, j_m\}$, $B = \{j_1, \dots, j_{l-1}\}$.

$\Rightarrow$ For every $C \subseteq \{1,\dots,n\}$, the product $\prod_{i \in C} b_i$ appears exactly once on the RHS(I). □

We now translate this to ~~the~~ number theory.

<u>Def</u> For $n>1$, denote by $\mathrm{lpf}(n)$ the least prime factor of n.

Let $z \geq 1$. (We'll only consider primes $p<z$ in our sieve.)

Let $\mathcal{D} \subseteq \mathcal{D}_0 := \left\{ d \geq 1 \text{ sqfree, only divisible by primes } < z \right\}$.

such that

a) $1 \in \mathcal{D}$

b) If $1 < d \in \mathcal{D}$, then $\frac{d}{\mathrm{lpf}(d)} \in \mathcal{D}$.

<u>Cor 11.3.2</u> Let $a_1, a_2, \ldots$ be multiplicative.

Then,

$$\prod_{p<z} (1+a_p) = \sum_{d \in \mathcal{D}} a_d + \sum_{\substack{d \in \mathcal{D}_0 \\ d \notin \mathcal{D} \\ \frac{d}{\mathrm{lpf}(d)} \in \mathcal{D}}} a_d \cdot \prod_{p<\mathrm{lpf}(d)} (1+a_p).$$

<u>Pf</u> Let $p_1 < \ldots < p_n$ be the prime numbers $<z$.

Let $S = \{ A \subseteq \{1, \ldots, n\} \mid \prod_{i \in A} p_i \in \mathcal{D} \}$.

Apply Lemma 11.3.1 to the numbers $a_{p_1}, \ldots, a_{p_n}$ and use that $a_{\prod_{i\in A} p_i} = \prod_{i \in A} a_{p_i}$. $\square$

Cor 11.3.3 ~~For~~ For any $b \in \mathbb{Z}$,

$$\sum_{\substack{d \in \mathcal{D}: \\ d \mid b}} \mu(d) + \sum_{\substack{d \in \mathcal{D}_0 \\ d \notin \mathcal{D} \\ \frac{d}{\mathrm{lpf}(d)} \in \mathcal{D} \\ d \mid b \\ p \nmid b \ \forall p < \mathrm{lpf}(d) \text{ prime}}} \mu(d) = \begin{cases} 1, & p \nmid b \ \forall p < z, \\ 0, & p \mid b \text{ for some } p < z. \end{cases}$$

Pf Take $a_d = \begin{cases} \mu(d), & d \mid b, \\ 0, & d \nmid b. \end{cases}$

This is multiplicative, with $a_p = \begin{cases} -1, & p \mid b, \\ 0, & p \nmid b. \end{cases}$

$$\prod_{p < q} (1 + a_p) = \begin{cases} 1, & p \nmid b \ \forall p < z, \\ 0, & p \mid b \text{ for some } p < z. \end{cases}$$

$\square$

Rmk Hence:

a) If $d \notin \mathcal{D}$, $\frac{d}{lpf(d)} \in \mathcal{D}$ implies $\mu(d) = -1$, we obtain an

upper bound sieve:

For any numbers $a_1, \dots, a_x \in \mathbb{Z}$,

$$\#\{n : a_n \text{ not div. by any } p < z\} \leq \sum_{d \in \mathcal{D}} \mu(d) \cdot \#\{n : d \mid a_n\}.$$

b) If --- implies $\mu(d) = +1$, we obtain a

lower bound sieve:

$$\#\{ \;---\; \} \geq \sum \cdots$$

Ex $\mathcal{D} = \mathcal{D}_0 \rightsquigarrow$ <u>basic sieve</u> (inclusion-exclusion)

Ex Let $r \geq 0$.

$\mathcal{D} := \{ d \in \mathcal{D}_0 : \underbrace{\nu(d)}_{\text{nr. of primes dividing } d} \leq r \} \rightsquigarrow$ <u>Brun sieve</u>

(inclusion-exclusion truncated after r^{+1} steps)

$\# = \#\{1 \mid a_n\} - \sum_p \#\{p \mid a_n\} + \sum_{p\neq q} \#\{pq \mid a_n\} \mp \dots$

(upper bound if r is even,
lower bound if r is odd)

Ex Let $\beta \geq 1$, $D \geq 1$

$$\mathcal{D}^+_{\beta,D} := \left\{ p_1 \cdots p_r \;\middle|\; \begin{array}{l} p_1 < \dots < p_r \leq z \text{ prime}, \\ \cancel{p_1^{\beta+1} p_2 \cdots p_r \leq D \text{ if } r \text{ is odd}} \\ \cancel{p_2^{\beta+1} p_3 \cdots p_r \leq D \text{ if } r \text{ is even}} \\ p_k^{\beta+1} p_{k+1} \cdots p_r \leq D \quad \forall 1 \leq k \leq r \text{ with } r-k \text{ even} \end{array} \right\}$$

(upper bound)

$$D^-_{\beta,D} := \left\{ p_1 \cdots p_r \;\middle|\; \begin{array}{l} p_1 < \dots < p_r \leq z \text{ prime}, \\ \cancel{p_1^{\beta+1} p_2 \cdots p_r \leq D \text{ if } r \text{ is even}}, \\ \cancel{p_2^{\beta+1} p_3 \cdots p_r \leq D \text{ if } r \text{ is odd}} \\ p_k^{\beta+1} p_{k+1} \cdots p_r \leq D \quad \forall 1 \leq k \leq r \text{ with } r-k \text{ odd} \end{array} \right\}$$

(lower bound)

$\rightsquigarrow$ <u>beta sieve</u> / <u>Rosser–Iwaniec sieve</u>

<u>Note:</u> $d \in \mathcal{D}^{\pm}_{\beta,D} \Rightarrow d \leq D$

~~Note: $d = p_1 \cdots p_r \in \mathcal{D}^{\pm}_{\beta,D}$ with $p_1 < \dots < p_r \leq z$ prime~~

We'll now analyse the main term in the beta sieve.

Def Let $\kappa > 0$. A multiplicative sequence $a_1, a_2, \ldots$ of real numbers with $0 \le a_p < 1$ is of <u>sieve dimension $\le \kappa$</u> if $V(w) := \prod_{p<w} (1-a_p)$

satisfies

$$\frac{V(z)}{V(w)} \gg \left(\frac{\log z}{\log w}\right)^{-\kappa} \quad \text{for } 2 \le w \le z.$$

all

Main example:

Lemma 1.3.4 If $0 \le a_p < 1$, $a_p \le \frac{\kappa}{p}$ for all suff. large p,

then $a_1, a_2, \ldots$ is of sieve dimension $\le \kappa$.

Pf Assume $a_p \le \frac{\kappa}{p}$ for $p \ge T$.

$$\frac{V(z)}{V(w)} = \prod_{w \le p < z} (1-a_p)$$

$$\gg \prod_{\substack{w \le p < z: \\ T \le p, \\ \kappa \le p}} (1-a_p)$$

$$\ge \prod_{\substack{w \le p < z: \\ \kappa \le p}} \left(1 - \frac{\kappa}{p}\right)$$

$$\gg \prod_{w \le p < z} \left(1 - \frac{1}{p}\right)^{\kappa}$$

$$= \left(\frac{\prod_{p<z} (1-\frac{1}{p})}{\prod_{p<w} (1-\frac{1}{p})} \right)^{\kappa} \asymp \left(\frac{\log z}{\log w}\right)^{-\kappa} \quad \text{for large } w, z.$$

PNT

$\square$