Thm 11.2.2 (Selberg sieve) Let $x, z \geq 1$.

Let $a_1, a_2, \ldots$ be a sequence of integers. Let $b_1, b_2, \ldots > 0$ be multiplicative and

Assume that $\#\{n \leq x : d \mid a_n\} = \frac{x}{b_d} + R_d$ for all $d \geq 1$.

Then,

$$\#\left\{n \leq x : \begin{array}{c} a_n \text{ not divisible by} \\ \text{any } p \leq z \end{array}\right\} \leq \frac{x}{U(z)} + O\Big(\sum_{d_1, d_2 \leq z} |R_{\mathrm{lcm}(d_1, d_2)}|\Big),$$

where $$U(z) = \sum_{\substack{d \leq z \\ \text{sqfree}}} \frac{1}{c_d}$$

with $b = c * 1$

$\left(\Leftrightarrow b_n = \sum_{d \mid n} c_d \Leftrightarrow c = b * \mu \Leftrightarrow c_n = \sum_{d \mid n} b_d \, \mu\left(\frac{n}{d}\right)\right)$.

Pf like the pf of Thm 11.2.1. $\square$

Cor 11.2.3 The number of twin primes $p, p+2 \le x$ is

$$\ll \frac{x}{(\log x)^2}.$$

Pf (sketch) ~~Take $a_n = n(n+2)$.~~

$$\#\{p, p+2 \text{ prime} : z < p < x\}$$

$$\le \#\{n \le \tfrac{x}{2}+10 : (2n+1)(2n+3) \text{ not div. by any prime } q \le z\}$$

Take $a_n = (2n+1)(2n+3)$

$$\#\{n \le x : d \mid a_n\}$$

$$= \sum_{\substack{d_1, d_2 \ge 1: \\ d = d_1 d_2, \\ \gcd(d_1, d_2) = 1}} \#\{n \le x : \underbrace{d_1 \mid 2n+1,\ d_2 \mid 2n+3}_{\Leftrightarrow\, n \equiv \dots \bmod d_1 d_2}\}$$

$$= \begin{cases} \sum\limits_{\dots} \frac{x}{d_1 d_2} + O(1) = \frac{2^{\nu(d)}}{d} \cdot x + O(2^{\nu(d)}), & d \text{ odd}, \\ 0, & d \text{ even}. \end{cases}$$

$$\rightsquigarrow b_d = \begin{cases} \frac{d}{2^{\nu(d)}}, & d \text{ odd} \\ \text{"}\infty\text{"}, & d \text{ even} \end{cases}$$

$$R_d = O(2^{\nu(d)})$$

$$c_d = \sum_{e \mid d} \frac{e}{2^{\nu(e)}} \cdot \mu\left(\frac{d}{e}\right)$$

$$U(z) = \sum_{\substack{d \leq z \\ \text{sqfree} \\ \text{odd}}} \frac{1}{c_d} \geq \sum_{\substack{d \leq z \\ \text{sqfree} \\ \text{odd}}} \frac{d}{2^{\nu(d)}}$$

Let $H(z) = \sum_{\substack{d \leq z \\ \text{sqfree} \\ \text{odd}}} 2^{\nu(d)}$. We have $H(z) \asymp z \log z$ by Wiener–Ikehara.

By Abel summation,

$$\sum_{\substack{d \leq z \\ \text{sqfree} \\ \text{odd}}} \frac{d}{2^{\nu(d)}} \; - \underbrace{\int_{1/2}^{z} \underbrace{\frac{H(t)}{t^2}}_{\asymp \frac{\log t}{t}} dt}_{\asymp (\log z)^2} = \underbrace{\left[\frac{H(t)}{t} \right]_{t=1/2}^{z}}_{\asymp \log z},$$

so $U(z) \gg (\log z)^2$.

Also, $$\sum_{d_1, d_2 \leq z} 2^{\nu(\mathrm{lcm}(d_1, d_2))} \leq \sum_{d_1, d_2 \leq z} 2^{\nu(d_1)} \cdot 2^{\nu(d_2)} = \Big(\sum_{d \leq z} 2^{\nu(d)} \Big) = \underbrace{H(z)^2}_{\asymp (z \log z)^2}.$$

Summary:

$$\# \{\text{twin primes} \leq x\} \ll z + \frac{x}{(\log z)^2} + z^2 (\log z)^2$$

For $z = x^{1/4}$, the RHS is $\asymp \frac{x}{(\log x)^2}$. $\square$

Basic heuristic

- The set of primes behaves like a random subset of $\mathbb{Z}_{\geq 2}$ which contains n with prob. $\frac{1}{\log n}$.

$\rightsquigarrow$ (expected nr. of twin primes $\leq x$)

$$\approx \sum_{n \leq x} \frac{1}{(\log n)(\log(n+2))} \sim \frac{x}{(\log x)^2}.$$

(This heuristic also suggests that there are ∞ many pairs of primes $p, p+1, \ldots$)

Refined heuristic

Fix $z \geq 1$, and let $K_z = \prod_{p \leq z} p$.

The set of primes behaves like a random subset of $\mathbb{Z}_{\geq 2}$ which contains n with prob.

$$\begin{cases} 0, & \gcd(n, K_z) > 1, \\ \dfrac{K_z}{\varphi(K_z) \log n}, & \gcd(n, K_z) = 1. \end{cases}$$

$$\frac{K_z}{\varphi(K_z)\log n} = \Big(\prod_{p \leq z} \underbrace{\frac{p}{p-1}}_{= \frac{1}{1-\frac{1}{p}}}\Big) \cdot \frac{1}{\log n}$$

("The larger z, the better the heuristic.")

$\rightsquigarrow$ (expected nr. of twin primes $\leq x$)

$$\approx \sum_{\substack{n \leq x; \\ \gcd(n, K_z) = 1 \\ \gcd(n+2, K_z) = 1}} \Big(\prod_{p \leq z} \frac{1}{1-\frac{1}{p}} \cdot \frac{1}{\log n}\Big)^2 \approx \frac{\#\{\text{good res. cl. mod } K_z\}}{K_z} \cdot x \cdot \Big(\prod_{p \leq z} \frac{1}{1-\frac{1}{p}} \cdot \frac{1}{\log x}\Big)^2$$

$$= \cancel{}\ \frac{1}{2} \cdot \prod_{2<p\le z} \left(1-\frac{2}{p}\right) \cdot 4 \cdot \prod_{2<p\le z} \frac{1}{\left(1-\frac{1}{p}\right)^2} \cdot \frac{x}{(\log x)^2}$$

$$= \underbrace{2 \cdot \prod_{2<p\le z} \left(1-\frac{1}{(p-1)^2}\right)}_{\xrightarrow{z\to\infty}\ C} \cdot \frac{x}{(\log x)^2}$$

$\leadsto$ heuristic: $\#(\text{twin primes} \le x) \sim 2C \cdot \frac{x}{(\log x)^2}$,

(Hardy–Littlewood)

References: Murty, Montgomery-Vaughan,

Terence Tao's blog: {254A notes 4: some sieve theory},

Friedlander-Iwaniec: Opera de Cribro