Similarly:

Thm 10.9 There exists $C>0$ such that for all a, q with $\gcd(a,q)=1$ and all $x > e^{C(\log q)^2}$, we have

$$\sum_{\substack{n\le x:\\ n\equiv a \bmod q}} \Lambda(n) = \frac{x}{\varphi(q)} + O\left(x\, e^{-c\sqrt{\log x}}\right)$$ if no char. $\chi \bmod q$ has a Siegel zero.

and

$$\sum_{\substack{n\le x:\\ n\equiv a \bmod q}} \Lambda(n) = \frac{x}{\varphi(q)} - \frac{\chi(a)x^{\beta}}{\beta\varphi(q)} + O(\dots)$$ if some char. $\chi \bmod q$ has a Siegel zero at β.

Rmk There can be at most one char. mod q with a Siegel zero! (HW)

Thm 10.10 Assuming the Generalized Riemann Hypothesis, for all a, q with $\gcd(a,q)=1$ and all $x \ge q$, we have

$$\sum_{\substack{n\le x:\\ n\equiv a \bmod q}} \Lambda(n) = \frac{x}{\varphi(q)} + O\left(x^{1/2}(\log x)^2\right).$$

11. Sieves

11.1. Basic sieve

Def An integer n is squarefree if it is not divisible by p^2 for any prime p.

Thm 11.1.1 We have

$$\#\{1 \le n \le x : n \text{ squarefree}\} = \frac{1}{\zeta(2)} \cdot x + O(x^{1/2}).$$

Rmk $\#\{\dots\} \sim \frac{1}{\zeta(2)} \cdot x$ for large x, follows from $\mu * 1_{\text{square}} = 1_{\text{squarefree}}$ and Wiener–Ikehara.

Pf Recall: $\mu * 1 = \delta$, so $\sum_{d \mid m} \mu(d) = \begin{cases} 1, & m = 1, \\ 0, & m \ge 2. \end{cases}$ (I)

$$\Rightarrow \sum_{\substack{d \ge 1: \\ d^2 \mid n}} \mu(d) = \begin{cases} 1, & n \text{ squarefree}, \\ 0, & \text{otherwise} \end{cases}$$

Take the largest m s.t. $m^2 \mid n$. $\Rightarrow (d^2 \mid n \Leftrightarrow d \mid m)$

$$\Rightarrow \#\{x \le n \text{ squarefree}\}$$

$$= \sum_{1 \le n \le x} \sum_{\substack{d \ge 1: \\ d^2 \mid n}} \mu(d) = \sum_{1 \le d \le x^{1/2}} \mu(d) \sum_{\substack{1 \le n \le x: \\ d^2 \mid n}} 1$$

$$= \sum_{1 \le d \le x^{1/2}} \mu(d) \cdot \left\lfloor \frac{x}{d^2} \right\rfloor$$

$$= \sum_{1 \le d \le x^{1/2}} \frac{\mu(d)}{d^2} \cdot x + O(x^{1/2}) \qquad \left(\frac{\mu(d)}{d^2} \ll \frac{1}{d^2}\right)$$

$$= \sum_{d \ge 1} \frac{\mu(d)}{d^2} \cdot x + O(x^{1/2})$$

$$= \frac{1}{\zeta(2)} \cdot x + O(x^{1/2}).$$

□

Rmk another way to look at it:

$\mathbb{P}(p^2 \nmid n : n \in \mathbb{Z} \text{ random}) = 1 - \frac{1}{p^2}$

By the Chinese Remainder Thm, these events $(p^2 \nmid n)$ are independent for finitely many distinct primes $p_1, \dots, p_u$.

If they were indep. for all p, we'd conclude that $\mathbb{P}(n \text{ squarefree}) = \prod_p \mathbb{P}(p^2 \nmid n) = \prod_p (1 - \frac{1}{p^2}) = \frac{1}{\zeta(2)}$,

$\rightsquigarrow \#\{1 \le n \le X \text{ squarefree}\} \sim \frac{1}{\zeta(2)} \cdot X$.

Rmk More generally, it is conjectured that for any polynomial $f(x) \in \mathbb{Z}[X]$, we have

$$\#\{1 \le n \le X : f(n) \text{ squarefree}\} \sim \prod_p \frac{\#\{a \in \mathbb{Z}/p^2\mathbb{Z} : p^2 \nmid f(a)\}}{p^2} \cdot X.$$

This is only known in some special cases, e.g. $\deg(f) \le 3$, or assuming the ABC conjecture.
(case $\deg = 2$ easy; $\deg = 3$ due to Hooley)

In general, we only know $\limsup_{X \to \infty} \frac{LHS}{RHS} \le 1$.

("sieves are better at upper bounds")

Thm 11.1.2 For $K \geq 1$ and any $x \geq 1$, we have

$$\#\{1 \leq n \leq x \text{ not divisible by any } p \leq x\}$$

$$= \prod_{p \mid K} \left(1 - \frac{1}{p}\right) \cdot x + O\left(2^{\nu(K)}\right),$$

where $\nu(K) =$ nr. of primes dividing K.

Pf $\#\{\dots\} = \sum_{d \mid K} \mu(d) \cdot \#\{1 \leq n \leq x : d \mid n\}$

$$= \sum_{d \mid K} \mu(d) \left(\frac{x}{d} + O(1)\right)$$

$$= \underbrace{\sum_{d \mid K} \frac{\mu(d)}{d}}_{\prod_{p \mid K}\left(1 - \frac{1}{p}\right)} \cdot x + O\Big(\underbrace{\#\{d \mid K \text{ sqfree}\}}_{2^{\nu(K)}}\Big)$$

□

11.2. Selberg sieve

Thm 11.2.1 Let $x, z \geq 1$. Then,

$$\pi(x,z) := \#\{1 \leq n \leq x \text{ not divisible by any } p \leq z\}$$
$$\leq \frac{x}{V(z)} + O(z^2)$$

with $V(z) := \sum_{d \leq z} \frac{\mu(d)^2}{\phi(d)} = \sum_{\substack{d \leq z \\ \text{sqfree}}} \frac{1}{\phi(d)}$.

(ϕ: Euler's totient function)

[illegible]

Rmk $\sum_{\substack{d \geq 1 \\ \text{sqfree}, \\ \text{only div. by primes} \\ p \nmid \leq z}} \frac{1}{\phi(d)} = \prod_{p \leq z} \left(1 + \frac{1}{\phi(p)}\right) = \prod_{p \leq z} \left(1 + \frac{1}{p-1}\right) = \prod_{p \leq z} \frac{1}{1 - \frac{1}{p}}$

(ϕ mult.)

$$= \frac{1}{\prod_{p \leq z} \mathbb{P}(p \nmid n : n \in \mathbb{Z} \text{ random})}$$

Pf Let $\lambda_1, \lambda_2, \dots$ be real numbers with $\lambda_1 = 1$ and $\lambda_n = 0$ for $n > z$.

Let $P_z := \prod_{p \le z} p$.

Note: n not div. by any $p \le z \iff \gcd(n, P_z) = 1$.

$$\Rightarrow \quad \pi(x,z) \le \sum_{1 \le n \le x} \Big(\underbrace{\sum_{d \mid \gcd(n, P_z)} \lambda_d}_{\substack{=1 \text{ if } \gcd = 1 \\ \ge 0 \text{ always}}} \Big)^2 \qquad (I)$$

$$= \sum_{\substack{d_1, d_2 \le z \\ (\Rightarrow d_1 d_2 \mid P_z)}} \lambda_{d_1} \lambda_{d_2} \, \#\{1 \le n \le x : d_1, d_2 \mid n\}$$

$$= \sum_{d_1, d_2 \le z} \lambda_{d_1} \lambda_{d_2} \left(\frac{x}{\mathrm{lcm}(d_1, d_2)} + O(1) \right) \qquad (II)$$

Note: (I) is an equality if we choose $\lambda_d = \mu(d)$.

We will now choose the numbers $\lambda_1, \dots, \lambda_z$ so that the quadratic form

$Q(\lambda) := \sum_{d_1 d_2 \le z} \frac{\lambda_{d_1} \lambda_{d_2}}{\mathrm{lcm}(d_1, d_2)}$ becomes as small as possible.

$$Q(\lambda) = \sum \frac{\lambda_{d_1}}{d_1} \cdot \frac{\lambda_{d_2}}{d_2} \cdot \gcd(d_1, d_2)$$

$$= \sum_{e \le z} \phi(e) \cdot \Big(\underbrace{\sum_{\substack{d \le z : \\ e \mid d}} \frac{\lambda_d}{d}}_{=: \nu_e} \Big)^2 = \sum_{e \le z} \phi(e) \nu_e^2$$

($\uparrow$ $\phi * 1 = \mathrm{Id}$, so $\sum_{e \mid t} \phi(e) = t$)

(diagonalization of Q)

~~We have~~ For any $d \geq 1$, we have

$$\sum_{k \geq 1} \mu(k) \nu_{dk} = \sum_{k \geq 1} \mu(k) \sum_{d \mid f \mid dk} \frac{\lambda_f}{f} = \sum_{d \mid f} \frac{\lambda_f}{f} \underbrace{\sum_{k \mid \frac{f}{d}} \mu(k)}_{\substack{1 \text{ if } \frac{f}{d} = 1 \\ 0 \text{ otherwise}}}$$

$$= \frac{\lambda_d}{d}.$$

Hence, $\lambda_1 = 1 \iff \sum_{k \geq 1} \mu(k) \nu_k = 1$

and $\lambda_n = 0 \ \forall n > z \iff \nu_n = 0 \ \forall n > z.$

$\Rightarrow$ We shall minimize $\sum_{e \leq z} \phi(e) \nu_e^2$

subject to the condition $g(\nu) = \sum_{1 \leq k \leq z} \mu(k) \nu_k = 1.$

Lagrange multipliers tells us to look at the points $(\nu_k)_k$ where $\sum \mu(e) \nu_e = 1$ and for some $\tau \in \mathbb{R}$, we have $\frac{\partial Q}{\partial \nu_e} = \tau \cdot \frac{\partial g}{\partial \nu_e}$ for all $1 \leq e \leq z$.

$$\frac{\partial Q}{\partial \nu_e} = 2\phi(e)\nu_e \qquad \frac{\partial g}{\partial \nu_e} = \mu(e)$$

$$\rightsquigarrow \nu_e = \frac{\tau \mu(e)}{2\phi(e)} \rightsquigarrow 1 \overset{!}{=} \sum \mu(e) \nu_e = \sum \frac{\tau \mu(e)^2}{2\phi(e)} = \frac{\tau}{2} \cdot V(z)$$

$\Rightarrow \frac{\tau}{2} = \frac{1}{V(z)}$, so $\nu_e = \boxed{\frac{1}{V(z)} \cdot \frac{\mu(e)}{\phi(e)}}$, so

$$Q = \sum_e \phi(e) \nu_e^2 = \frac{1}{V(z)^2} \cdot \sum_e \frac{\mu(e)^2}{\phi(e)} = \frac{1}{V(z)}.$$

(Indeed, for this choice of ν_e, we have $\lambda_1 = 1$ etc.)

~~Also, $\frac{\lambda_d}{d} = \sum_{k\geq 1} \mu(k)\nu_{dk} = \sum_{k\geq 1} \mu(k)\cdot\frac{1}{V(z)}\cdot\frac{\mu(dk)}{\phi(dk)}$~~

~~$\Rightarrow \pi(x,z) \leq \frac{x}{V(z)} + O\Big(\sum_{d_1 d_2 \leq z} |\lambda_{d_1}\lambda_{d_2}|\Big)$~~

Also, $\left|\frac{\lambda_d}{d}\right| = \left|\sum_{k\geq 1}\mu(k)\nu_{dk}\right| = \left|\sum_{1\leq k\leq z:\, dk\leq z} \mu(k)\cdot\frac{1}{V(z)}\cdot\frac{\mu(dk)}{\phi(dk)}\right|$

$\leq \sum_{1\leq k\leq z:\, \text{sqfree},\, \gcd(d,k)=1,\, dk\leq z} \frac{1}{V(z)}\cdot\frac{1}{\phi(d)\phi(k)}$,

so $|\lambda_d|\cdot V(z) \leq \Big(\sum_{\substack{1\leq k\leq z\\ \text{sqfree}:\\ dk\leq z\\ \gcd(d,k)=1}} \frac{1}{\phi(k)}\Big)\cdot\frac{d}{\phi(d)} = \sum_{1\leq k\leq z}\frac{\mu(k)^2}{\phi(k)} = V(z)$.

$\frac{d}{\phi(d)} \leq \sum_{e\mid d}\frac{\mu(e)^2}{\phi(e)}$

$\Rightarrow |\lambda_d| \leq 1$.

Plugging into (★):

$\pi(x,z) \leq \frac{x}{V(z)} + O(z^2)$. □