Thm 10.5 (PNT with error term)

There is a constant $C>0$ s.t.

$$\sum_{n\leq x} \Lambda(n) = x + O\left(x e^{-C\sqrt{\log x}}\right) \text{ for large } x.$$

$$\left(= \sum_{p\leq x} \log p + O\left(x^{1/2}(\log x)^2\right)\right)$$

Rmk For any $k, \varepsilon > 0$,

$(\log x)^k << e^{C\sqrt{\log x}} << x^{\varepsilon}$ for large x.

Pf Let $c = 1 + \frac{1}{\log x}$. Let D be the constant from Thm 9.2.6 so $\zeta(s)$ has no zero ρ with

$$\mathrm{Re}(\rho) > 1 - \frac{D}{\log(|\mathrm{Im}\,\rho| + 2)}.$$

Let $\mathcal{C}$ be the boundary of

$$\left\{ s \in \mathbb{C} : |\mathrm{Im}\, s| \le T,\ 1 - \frac{D}{2\log(|\mathrm{Im}\, s| + 2)} \le \mathrm{Re}(s) \le c \right\}.$$

By Lemma 10.4,

$$\sum_{n \le x} \Lambda(n) = \frac{1}{2\pi i} \int_{\text{right edge}} \underbrace{-\frac{\zeta'}{\zeta}(s) \frac{x^s}{s}}_{\substack{\text{pole of order 1} \\ \text{and residue } x \\ \text{at } s=1}} ds + O\left(\boxed{\frac{x (\log x)^2}{T}} \right).$$

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \cdots = x \quad \text{by the residue theorem.}$$

By Lemma 9.2.5, on ℓ, we have $\frac{\zeta'}{\zeta}(s) \ll (\log T)^2$.

$$\Rightarrow \int_{top} \cdots \ll \int_{top} (\log T)^2 \cdot \frac{x}{T}\, ds \ll \boxed{\frac{x(\log T)^2}{T}}.$$

$$\int_{left} \cdots \ll \int_{left} (\log T)^2 \cdot \frac{x^{\mathrm{Re}(s)}}{|s|}\, |ds|$$

$$\ll \int (\log T)^2 \cdot \frac{x^{1-\frac{D}{2\log T}}}{|\mathrm{Im}(s)|+1}\, |ds|$$

$$\ll x^{1-\frac{D}{2\log T}} \cdot \Big(1 + \underbrace{\int_1^T \frac{1}{y}\, dy}_{\log T}\Big)(\log T)^2$$

$$\ll \boxed{\frac{x(\log T)^3}{x^{D/2\log T}}}.$$

To optimize the error term, solve

$$T = x^{D/2\log T} \quad \text{for } T:$$

$$\Updownarrow$$

$$\log T = \frac{D\log x}{2\log T} \iff \log T = \sqrt{\frac{D\log x}{2}}.$$

$$\Downarrow$$

$$\text{error term } \frac{x(\log T)^3}{e^{\sqrt{\frac{D}{2}\log x}}}.$$

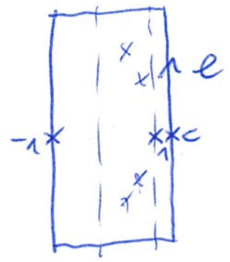
$$\ll x e^{-E\sqrt{\log x}} \text{ for any } E < \sqrt{\tfrac{D}{2}}.$$

□

Rmk Assuming the RH, we can get a better error bound! (E.g. use a larger region...)

Thm 10.6 Let x, T be large. Then, we have
$$\sum_{n\le x}\Lambda(n) = x - \sum_{\rho:|\operatorname{Im}\rho|<T}\frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) + O\left(\frac{x(\log T)^2}{T} + \frac{(\log T)^2}{x}\right).$$
and $x-\frac{1}{2}\in\mathbb{Z}$.

Pf Use the boundary ℓ of $[-1,c]+[-T,T]\cdot i$.



$$\frac{1}{2\pi i}\int_\ell \underbrace{-\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}}_{\substack{\text{pole of order 1 and residue } x \text{ at } s=1 \\ \text{pole of order 1 and residue } \frac{x^\rho}{\rho} \text{ at } s=\rho \\ \text{pole of order 1 and residue } -\frac{\zeta'}{\zeta}(0) \text{ at } s=0}} ds = x - \sum_{\substack{\rho: \\ |\operatorname{Im}\rho|<T}}\frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0)$$

$$\int_{\text{left}} \cdots \ll \frac{(\log T)^2}{x}$$

$-\frac{\zeta'}{\zeta}(s) \ll \log|s|$ for large $|s|$ with $\operatorname{Re}(s)=-1$

$\int_{\text{top}} \cdots$ is problematic because there might be a root very close to the contour.

But we know that there are $\ll \{\log T\}$ roots with $|\operatorname{Im}\rho - T| < 1$ according to Lemma 9.2.5.

$\Rightarrow$ For some constant $\delta > 0$ (indep. of T), there is some $T' = T + O(1)$ s.t. there are no roots with $|\operatorname{Im}\rho - T'| < \frac{\delta}{\log T}$.

Replace T by T' in the above computation. This changes $\sum \frac{x^\rho}{\rho}$ by $\ll (\log T) \cdot \frac{x}{T}$.

By the pf of Lemma 9.2.5,

$$\frac{\zeta'}{\zeta}(s) \ll (\log T)^2 \text{ on the top contour.}$$

$$\Rightarrow \int_{\text{top}} \cdots \ll \frac{x(\log T)^2}{T}. \qquad \square$$

Cor 10.7 Assume the Riemann Hypothesis. Then,

$$\sum_{n \le x} \Lambda(n) = x + O(x^{1/2}(\log x)^2).$$

Pf Take $T = x$.

$$\sum_{\substack{\rho: \\ |\operatorname{Im}\rho| < T}} \frac{x^\rho}{\rho} \ll x^{1/2} \cdot \sum \frac{1}{\rho} \ll x^{1/2}(\log x)^2.$$

$\operatorname{Re}(\rho) = \frac{1}{2}$

$\#\{\rho : |\operatorname{Im}\rho| < T\} \ll T \log T$ by Thm 9.2.4 and use Abel summation

$\square$

We can actually do even better:

Thm 10.8 Let $x-\frac{1}{2}\in\mathbb{Z}$ be large.

Then, $$\sum_{n\le x}\Lambda(n) = x - \sum_{\substack{\rho \text{ nontriv.} \\ \text{zero of } \zeta}} \frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log\left(1-\frac{1}{x^2}\right).$$

Idea of pf

Use a contour $[-U, c] + [-T, T]\cdot i$.

First, let $U \to \infty$, then $T \to \infty$.

$$\frac{1}{2\pi i}\int_{\ell} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds \longrightarrow x - \sum_{\substack{\rho \\ \text{zero} \\ \text{of } \zeta}} \frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0)$$

$$= x - \sum_{\substack{\rho \text{ nontriv.} \\ \text{zero of } \zeta}} \frac{x^\rho}{\rho} - \underbrace{\sum_{k=1}^{\infty} \frac{x^{-2k}}{-2k}}_{-\frac{1}{2}\log\left(1-\frac{1}{x^2}\right)} - \log(2\pi)$$

$\square$