Let $t = \mathrm{Im}(\rho)$ and $L = \log(q(|t|+2))$. Fix some $\delta > 0$.

For $\sigma > 1$, if $|t| \geq \frac{\delta}{\log q}$ or $\chi^2 \neq \chi_0$, then:
($\chi$ nonreal)

$$\mathrm{Re}\left(-\frac{L'}{L}(\sigma,\chi_0)\right) \leq \frac{1}{\sigma-1} + O(L)$$

$$\mathrm{Re}\left(-\frac{L'}{L}(\sigma+it,\chi)\right) \leq -\frac{1}{\sigma-\mathrm{Re}(\rho)} + O(L)$$

$$\mathrm{Re}\left(-\frac{L'}{L}(\sigma+2it,\chi^2)\right) \leq O(L) \quad \text{if } \chi^2 \neq \chi_0$$

$$\text{—''—} \quad \leq \mathrm{Re}\left(\frac{1}{\sigma+2it-1}\right) + O(L) \leq O(L) \quad \text{if } \chi^2 = \chi_0$$
($\chi$ real)

As before, it then follows that

$$\mathrm{Re}(\rho) \geq 1 - \frac{c}{L}. \qquad (c \text{ depends on } \delta.)$$

We now deal with the case $\chi^2 = \chi_0$ and $|\mathrm{Im}(\rho)| < \frac{\delta}{\log q}$

Clearly, $\mathrm{Re}\Big(\underbrace{-\frac{L'}{L}(\sigma,\chi_0)}_{\sum_{\dots} \frac{\Lambda(n)}{n^\sigma}} \underbrace{-\frac{L'}{L}(\sigma,\chi)}_{\sum_{\dots} \frac{\Lambda(n)\chi(n)}{n^\sigma}}\Big) \geq 0$ for any $\sigma > 1$.

We have

$$\operatorname{Re}\left(-\frac{L'}{L}(\sigma,\chi_0)\right) \leq \frac{1}{\sigma-1} + O(\log q)$$

$$\operatorname{Re}\left(-\frac{L'}{L}(\sigma,\chi)\right) \leq -\sum_{\rho} \operatorname{Re}\left(\frac{1}{\sigma-\rho}\right) + O(\log q)$$

$$\Rightarrow \quad \frac{1}{\sigma-1} - \sum_{\rho} \operatorname{Re}\left(\frac{1}{\sigma-\rho}\right) + O(\log q) \geq 0.$$

Take $\sigma = 1 + \frac{2\delta}{\log q}$

Then, for any $\rho$ with $|\operatorname{Im}(\rho)| < \frac{\delta}{\log q} = \frac{\sigma-1}{2}$

$$\operatorname{Re}\left(\frac{1}{\sigma-\rho}\right) = \frac{\sigma-\operatorname{Re}(\rho)}{|\sigma-\rho|^2} \geq \frac{\sigma-\operatorname{Re}(\rho)}{(\sigma-\operatorname{Re}(\rho))^2 + \left(\frac{\sigma-1}{2}\right)^2}$$

$$\geq \frac{4}{5(\sigma-\operatorname{Re}(\rho))}.$$

$$\Rightarrow \sum_{\substack{\rho:\\ |\operatorname{Im}(\rho)|<\frac{\delta}{\log q}}} \frac{4}{5(\sigma-\operatorname{Re}(\rho))} \leq \frac{\log q}{2\delta} + O(\log q)$$

If there are two $\rho$ with $|\operatorname{Im}(\rho)| < \frac{\delta}{\log q}$

and $\operatorname{Re}(\rho) > 1 - \frac{c}{\log q}$, then

$$2 \cdot \frac{4}{5\left(\frac{2\delta+c}{\log q}\right)} \leq \frac{\log q}{2\delta} + O(\log q).$$

For suff. small $\delta, c$, this is impossible because $\frac{2\cdot 4}{5\cdot 2} > \frac{1}{2}$.

Hence, $L(s,\chi)$ has at most one bad ~~real~~ zero $\rho$.

Since $\chi$ is real, this implies that $\rho$ is real. $\square$

# 10. Perron's formula

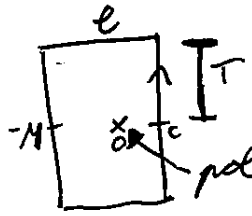
**Lemma 10.1** For $y \geq 0$, $c > 0$, we have

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{s}\,ds = \begin{cases} 0, & 0 \leq y < 1, \\ \frac{1}{2}, & y = 1, \\ 1, & y > 1. \end{cases}$$

Pf For $y=1$, $LHS = \lim\limits_{T\to\infty} \frac{1}{2\pi i}\int \frac{ds}{s} = \lim\limits_{T\to\infty}\frac{1}{2\pi i}\underbrace{\Big[\log s\Big]_{s=c-iT}^{c+iT}}_{\substack{\mathrm{Re}(-)=0 \\ \mathrm{Im}(-)\to\pi}} = \frac{1}{2}.$

For $y > 1$, the integrand $\frac{e^{s\log y}}{s}$ goes to 0 as $\mathrm{Re}(s) \to -\infty$.

$\rightsquigarrow$ Consider the boundary $\ell$ of the rectangle $[-M, c] + [-T, T]\cdot i$ in $\mathbb{C}$ for large $M>0$



pole of $\frac{y^s}{s}$ of residue 1

$$\underset{\substack{\uparrow \\ \text{Residue Theorem}}}{1} = \frac{1}{2\pi i}\int_\ell \frac{y^s}{s}\,ds = \int_{c-iT}^{c+iT}\frac{y^s}{s}\,ds + \underbrace{\int_{c+iT}^{-M+iT} \cdots}_{\mathcal{O}\left(\frac{e^{\mathrm{Re}(s)\log y}}{T}\right) \,\to\, \mathcal{O}\left(\frac{y^c}{T\log y}\right) \xrightarrow[T\to\infty]{} 0} + \underbrace{\int_{-M+iT}^{-M-iT} \underbrace{\cdots}_{\mathcal{O}\left(\frac{e^{-M\log y}}{M}\right)}}_{\xrightarrow{M\to\infty} 0 \text{ for fixed } T} + \underbrace{\int \cdots}_{\mathcal{O}\left(\frac{y^c}{T\log y}\right) \xrightarrow{T\to\infty} 0}$$

For $y<1$, use the rectangle $[c,M]+[-T,T]\cdot i$ for large $M$. □

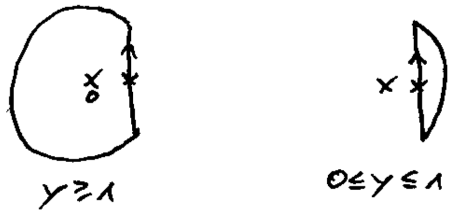
More precisely:

<u>Thm 10.2</u> We have

$$\left| \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{s}\,ds - \left\{\begin{matrix} 0 & \cdots \\ \frac{1}{2} & \cdots \\ 1 & \cdots \end{matrix}\right\} \right| \ll \min\Big(y^c, \underbrace{\frac{y^c}{T|\log y|}}_{(\infty \text{ for } y=1)}\Big).$$

<u>Pf</u> The second bound $\left(\cdots \ll \frac{y^c}{T|\log y|}\right)$ follows from the previous proof. The case $y=1$ is HW.

For the first bound $(\cdots \ll y^c)$, use the boundary of $\{s\in\mathbb{C} : |s|\le|c+iT|, \operatorname{Re}(s)\le c\}$ if $y\ge 1$ and of $\{s\in\mathbb{C} : |s|\le|c+iT|, \operatorname{Re}(s)\ge c\}$ if $0\le y\le 1$.



$y\ge 1$ $\qquad$ $0\le y\le 1$

On the arc, $\frac{y^s}{s} \ll \frac{y^c}{|c+iT|}$

(of length $O(c+iT)$)

□

**Cor 10.3** Let $c>0$.

Consider a Dirichlet series $D(a,s)=\sum a_n n^{-s}$ with abscissa of absolute convergence $\sigma_a < c$.

Then,

$$\sum_{n\le x} a_n = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} D(a,s)\,\frac{x^s}{s}\,ds$$

for any $x>0$ with $x\notin\mathbb{Z}$.
(otherwise, only count $a_x$ half)

Pf Since $\underset{\textstyle \sum_n a_n \cdot \frac{(x/n)^s}{s}}{\overset{\parallel}{D(a,s)\frac{x^s}{s}}}$ is uniformly convergent on the contour,

$$\frac{1}{2\pi i}\int D(a,s)\,\frac{x^s}{s}\,ds = \sum_n a_n \cdot \frac{1}{2\pi i}\int \frac{(x/n)^s}{s}\,ds$$

$$= \sum_{n\le x} a_n + O\Big(\underbrace{\sum_n |a_n| \left(\frac{x}{n}\right)^c}_{<\infty} \cdot \frac{1}{T\underbrace{\left|\log\frac{x}{n}\right|}_{\gg 1 \text{ for fixed } x\notin\mathbb{Z}}}\Big)$$

$\square$

We can again bound the error term. For example:

**Lemma 10.4** If $x - \frac{1}{2} \in \mathbb{Z}$, then

$$\left| \sum_{n \geq 1} \Lambda(n) - \frac{1}{2\pi i} \int_{c-iT}^{c+iT} -\frac{\zeta'}{\zeta}(s) \cdot \frac{x^s}{s} \, ds \right| \ll \frac{x (\log x)^2}{T}$$

for $c = 1 + \frac{1}{\log x}$.

$\uparrow$ (optimal choice!)

**Pf** Fix $\varepsilon > 0$ suff. small. We bound the error term from the pf of Cor 10.3:

$$\sum_{\substack{n \geq 1 \\ |\log \frac{x}{n}| \geq \varepsilon}} \Lambda(n) \left(\frac{x}{n}\right)^c \cdot \underbrace{\frac{1}{T \left|\log \frac{x}{n}\right|}}_{\ll \frac{1}{T}}$$

$$= -\frac{\zeta'}{\zeta}(c) \cdot \frac{x^c}{T} \underset{\substack{\uparrow \\ \text{only simple} \\ \text{pole at } s=1}}{\ll} \frac{1}{c-1} \cdot \frac{x^c}{T} = \frac{x (\log x) e}{T}$$

$$\sum_{\substack{n \geq 1 \\ |\log \frac{x}{n}| \leq \varepsilon}} \underbrace{\Lambda(n)}_{\leq \log n} \underbrace{\left(\frac{x}{n}\right)^c}_{\ll 1} \cdot \frac{1}{T \left|\log \frac{x}{n}\right|}$$

$$\left|\log \frac{1}{1 - (1 - \frac{n}{x})}\right| = \left| \sum_{k \geq 1} \frac{1}{k} \left(1 - \frac{n}{x}\right)^k \right| \underset{\substack{\uparrow \\ \varepsilon \text{ suff.} \\ \text{small}}}{\gg} \left|1 - \frac{n}{x}\right|$$

$$\ll \sum_{x e^{-\varepsilon} \leq n \leq x e^{\varepsilon}} (\log x) \cdot \frac{x}{T \underbrace{|n - x|}_{\in \frac{1}{2} + \mathbb{Z}}} \ll \frac{x (\log x)^2}{T}$$

$\square$