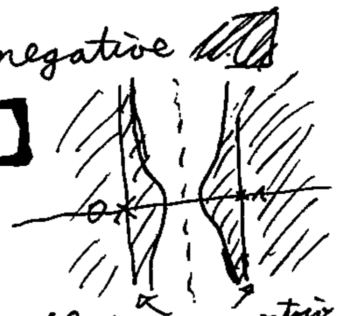
__Thm 9.2.6__ There is a constant $c>0$ such that $\zeta(s)$ has no (nontrivial) zero $\rho \in \mathbb{C}$ with $\mathrm{Re}(\rho) > 1 - \frac{c}{\log(|\mathrm{Im}(\rho)|+2)}$.

[For large $\mathrm{Im}(\rho)$, we could just write $\log|\mathrm{Im}(\rho)|$, but for small $\mathrm{Im}(\rho)$, that would be negative, for $\mathrm{Im}(\rho) \doteq 1$, it would be 0, etc. ...]



__Pf__ We saw in the pf of Thm 4.5 that

for any $\sigma > 1$ and $t \in \mathbb{R}$,

$$\mathrm{Re}\left(-3 \cdot \frac{\zeta'}{\zeta}(\sigma) - 4 \cdot \frac{\zeta'}{\zeta}(\sigma+it) - \frac{\zeta'}{\zeta}(\sigma+2it)\right) \geq 0. \qquad (I)$$

By Cor 9.2.3 and Cor 9.1.3, for $1 < \mathrm{Re}(s) < 2$,

$$\underbrace{\frac{1}{s}}_{\ll 1} + \frac{1}{s-1} + \underbrace{\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)}_{\ll \log(|\mathrm{Im}(s)|+2)} + \frac{\zeta'}{\zeta}(s) = B + \sum_\rho \Big(\underbrace{\frac{1}{s-\rho}}_{\mathrm{Re}(\cdot)>0} + \underbrace{\frac{1}{\rho}}_{\mathrm{Re}(\cdot)>0}\Big)$$

Let $t = \mathrm{Im}(\rho)$ and let $L = \log(|t|+2)$.

$\Rightarrow$ For $\sigma > 1$, if $|t| \geq 1$ (so $\frac{1}{\sigma+it-1} \ll 1$), then

$$\mathrm{Re}\left(-\frac{\zeta'}{\zeta}(\sigma)\right) \leq \frac{1}{\sigma-1} + O(L),$$

$$\mathrm{Re}\left(-\frac{\zeta'}{\zeta}(\sigma+it)\right) \leq -\frac{1}{\sigma-\mathrm{Re}(\rho)} + O(L),$$

$$\mathrm{Re}\left(-\frac{\zeta'}{\zeta}(\sigma+2it)\right) \leq O(L).$$

$$\underset{(\mathrm{I})}{\Longrightarrow} \quad \frac{3}{\sigma-1} - \frac{4}{\sigma-\mathrm{Re}(\rho)} + \mathcal{O}(L) \geq 0$$

Take $\sigma = 1 + \frac{\varepsilon}{L}$ for some small $\varepsilon < L$.

$$\Rightarrow \quad \frac{3}{\varepsilon} L - \frac{4L}{(1-\mathrm{Re}(\rho))L + \varepsilon} + \mathcal{O}(L) \geq 0$$

$$\Rightarrow \quad (1-\mathrm{Re}(\rho))L + \varepsilon \geq \frac{4}{\frac{3}{\varepsilon} + \mathcal{O}(1)}$$

$$\Rightarrow \quad (1-\mathrm{Re}(\rho))L \geq \varepsilon \cdot \left(\frac{4}{3+\mathcal{O}(\varepsilon)} - 1 \right)$$

For suff. small $\varepsilon > 0$, the RHS is $\geq$ some constant $c > 0$.

$$\Rightarrow \quad \mathrm{Re}(\rho) \geq 1 - \frac{c}{L}. \qquad \square$$

9.3. Dirichlet L-series

Lemma 9.3.1
For primitive characters χ mod $q > 1$, we have

$$-\frac{\xi'}{\xi}(s,\chi) = B_\chi + \sum_{\substack{\rho \text{ nontriv} \\ \text{zero of } L(s,\chi)}} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

Pf as for $\xi(s)$. $\square$

Lemma 9.3.2 We have $\mathrm{Re}\left(B_\chi + \sum_\rho \frac{1}{\rho}\right) = 0$.

Pf HW. $\square$

Lemma 9.3.3 If χ is the char. mod q induced by the char. χ' mod q' (with $q' \mid q$), then

$$\frac{L'}{L}(s,\chi) = \frac{L'}{L}(s,\chi') + O(\log q) \quad \text{if } \Re(s) > 1.$$

Pf ~~This follows from~~

$$L(s,\chi) = L(s,\chi') \cdot \prod_{\substack{p \mid q: \\ p \nmid q'}} \left(1 - \frac{\chi'(p)}{p^s}\right)$$

$$\Rightarrow \frac{L'}{L}(s,\chi) = \frac{L'}{L}(s,\chi') - \underbrace{\sum_{p \mid q} \underbrace{\sum_{k \geq 1} \frac{\chi'(p^k) \log p}{p^{ks}}}_{\ll \log p}}_{\ll \log q.}$$

$\square$

Thm 9.3.4

There is a constant $c>0$ such that for any character χ mod any q, $L(s,\chi)$ has no (nontriv.) zero $\rho \in \mathbb{C}$ with $\mathrm{Re}(\rho) > 1 - \frac{c}{\log(q(|\mathrm{Im}(\rho)|+2))}$,

except possibly one real zero $\rho \in \mathbb{R}$ if χ is real.

Pf: W.l.o.g. χ is primitive.
We've already proved the result for $q=1$, so assume $q>1$. ($\Rightarrow \chi \neq \chi_0$)

First attempt: Use the same strategy as before, replacing $\zeta(s)$ by $\prod_{\chi} L(s,\chi)$. This only proves the above statement with the constant c depending on q.

~~Let $t = \mathrm{Im}(\rho)$ and $L = \log(q(|t|+2))$.~~

Now, $3 + 4\cos\theta + \cos 2\theta \geq 0$ implies:
For any $\sigma > 1$ and $t \in \mathbb{R}$:

$$\mathrm{Re}\Big(-3\cdot \underbrace{\frac{L'}{L}(\sigma,\chi_0)}_{-\sum\limits_{\gcd(n,q)=1} \frac{\Lambda(n)}{n^\sigma}} - 4\cdot \underbrace{\frac{L'}{L}(\sigma+it,\chi)}_{-\sum\limits_{\dots} \frac{\Lambda(n)\chi(n)}{n^{\sigma+it}}} - \underbrace{\frac{L'}{L}(\sigma+2it,\chi^2)}_{-\sum\limits_{\dots} \frac{\Lambda(n)\chi(n)^2}{n^{\sigma+2it}}}\Big) \geq 0$$