Cor 9.1.2

$$\Gamma(s)^{-1} = e^{A+Bs}\cdot s\cdot \prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-s/n}$$

$$-\frac{\Gamma'}{\Gamma}(s) = B + \frac{1}{s} + \sum_{n=1}^{\infty}\left(\frac{1}{s+n} - \frac{1}{n}\right)$$

Pf $s^{-1}\Gamma(s)^{-1}$ has order 1 by Stirling's formula (for $\mathrm{Re}(s)\geq\frac{1}{2}$) and because $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$.

- Its zeros are $-1, -2, \ldots$ $\square$

Cor 9.1.3 If $\mathrm{Re}(s) > 0$, then

$$-\frac{\Gamma'}{\Gamma}(s) \ll \log|s| \quad \text{for large } |s|.$$

Pf $-\frac{\Gamma'}{\Gamma}(s) = O(1) + \sum_{n=1}^{\infty}\frac{-1}{n(1+\frac{n}{s})}$

Split up the sum:

for $n \leq 2|s|$: $\mathrm{Re}(1+\frac{n}{s}) \geq 1$, so $\Sigma(\ldots) \ll \Sigma \frac{1}{n} \ll \log|s|$

for $n \geq 2|s|$: $|1+\frac{n}{s}| \geq \frac{1}{2}|\frac{n}{s}|$, so $\Sigma(\ldots) \ll \Sigma \frac{|s|}{n^2} \ll \frac{|s|}{|s|} = 1$ $\square$

9.2. Riemann zeta function

Reminder: $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is hol. except for simple poles at $s = 0, 1$ and satisfies $\xi(s) = \xi(1-s)$. Its zeros are the nontrivial zeros of $\zeta(s)$.

Thm 9.2.1 The (entire) function $f(s) := s(s-1)\xi(s)$ has order 1.

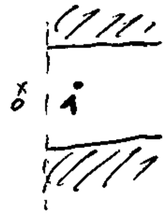
~~Pf By the functional equation $f(s) = f(1-s)$, it suffices to consider $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq \frac{1}{2}$.~~

Pf This follows from the functional equation $f(s) = f(1-s)$, and the following lemma, Stirling's approximation for $\Gamma(s/2)$, $\square$
(By the fct. eq., we only need to consider $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq \frac{1}{2}$)

Lemma 9.2.2 a) $\zeta(s) \geq 1$ for $s > 1$.

b) For any $\varepsilon > 0$: $\zeta(s) \ll_\varepsilon 1$ if $\mathrm{Re}(s) > 1 + \varepsilon$

c) $\zeta(s) \ll |\mathrm{Im}(s)|$ if $\mathrm{Re}(s) \geq \frac{1}{2}$, $|\mathrm{Im}(s)| \geq 1$.



Pf a) clear, b) clear from $\zeta(s) = \sum \frac{1}{n^s}$.
c) Use Euler-Maclaurin: W.l.o.g. $\mathrm{Re}(s) < 2$. As in the pf. of Thm 3.2

$$\zeta(s) - 1 = \sum_{n=2}^{\infty} \frac{1}{n^s} = \underbrace{\frac{1}{s-1} - \frac{1}{2} + \int_1^\infty \underbrace{B_1(t)}_{\ll 1} \cdot \underbrace{\frac{s}{t^{s+1}}}_{\ll \frac{|s|}{t^{\mathrm{Re}(s)+1}}}\, dt}_{\ll |\mathrm{Im}(s)|}.$$

$\square$

Cor 9.2.3

We can write

$$s(s-1)\,\xi(s) = e^{A+Bs}\cdot \prod_{\substack{\rho \text{ nontriv.} \\ \text{zero of } \xi(s)}} \left(1-\frac{s}{\rho}\right)e^{s/\rho},$$

$$\frac{1}{s}+\frac{1}{s-1}+\frac{\xi'}{\xi}(s) = B + \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right).$$

Rmk We have $\lim_{s\to 1} s(s-1)\xi(s) = \lim_{s\to 1}(s-1)\xi(s) = 1$, so $A=0$.

$\boxed{\Gamma(1/2)=\sqrt{\pi}}$

<u>Thm 9.2.4</u> The number $N(T)$ of nontriv. zeros of $\zeta(s)$ with $0 \le \mathrm{Im}(s) \le 2\pi T$ is

$$T \log T - T + O(\log T) \text{ for large } T.$$

<u>Rmk</u> <u>Informally</u>, the nr. of nontriv. zeros with $2\pi T \le \mathrm{Im}(s) \le 2\pi(T+1)$ is

$$\approx \frac{d}{dT}(T \log T - T) = \log T.$$

~~We first show:~~
~~Lemma 9.2.5~~

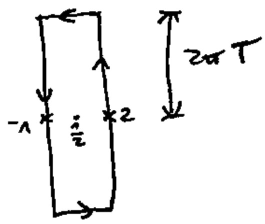
<u>Rmk</u> The Thm implies that this nr. is $\ll \log T$.

Pf of Thm 9.2.4

Let $\mathcal{C}$ be the ccw boundary of

$$[-1,2]\times[-2\pi T, 2\pi T].$$

W.l.o.g. no zeros on $\mathcal{C}$.

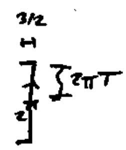


$N(T)+O(1)$

$= \frac{1}{2}$(nr. of zeros in the rectangle, with mult. minus nr. of poles of $\Xi(s)$)

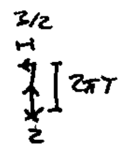
$$= \frac{1}{2}\cdot\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{\xi'}{\xi}(s)\,ds$$

$$= \frac{1}{2\pi i}\oint_{\mathcal{D}} \frac{\xi'}{\xi}(s)\,ds \qquad \text{for } \mathcal{D} \text{ the right half of } \mathcal{C}$$



$$\frac{\xi'}{\xi}(1-s) = -\frac{\xi'}{\xi}(s)$$

$$= \frac{1}{\pi}\,\mathrm{Im}\oint_{\mathcal{E}} \frac{\xi'}{\xi}(s)\,ds \qquad \text{for } \mathcal{E} \text{ the top half of } \mathcal{D}$$



$$\frac{\xi'}{\xi}(\bar{s}) = \overline{\frac{\xi'}{\xi}(s)}$$

Split up $\mathcal{E}$ into the vertical part $\mathcal{E}_1$ and the horizontal part $\mathcal{E}_2$.

Let's first deal with $\mathcal{E}_1$.

Let $f(s) = \pi^{-s}\Gamma(s)$ so that $\xi(s) = f(s)\zeta(s)$.

$$\rightarrow \frac{\xi'}{\xi}(s) = \frac{1}{2}\frac{f'}{f}\left(\frac{s}{2}\right) + \frac{\zeta'}{\zeta}(s).$$

~~Key (Stirling's formula~~

$$\oint_{\mathcal{E}_1} \frac{1}{2}\frac{f'}{f}\left(\frac{s}{2}\right) ds = \oint_{\mathcal{E}_1/2} \frac{f'}{f}(s)\, ds \overset{!}{=} \oint_{\mathcal{E}_1/2} d\,\overbrace{\log f(s)}^{\text{well-def. in } \{\mathrm{Re}(s)>0\}.}$$

$$= \log f\left(\frac{2+2\pi i T}{2}\right) - \log f\left(\frac{2}{2}\right)$$

$$\underset{\substack{\uparrow \\ \text{Stirling}}}{=} -(\cancel{1}+\pi i T)\log\pi + (\cancel{1}+\pi i T)\underbrace{\log(\cancel{1}+\pi i T)}_{\log(\pi T)+\frac{\pi}{2}i} - (\cancel{1}+\pi i T) + O(\log T)$$

$$= \pi i T \log T - \pi i T - \frac{\pi^2}{2}\cdot T + O(\log T)$$

has imaginary part $\pi T\log T - \pi T$. This gives the main term.

If $\mathrm{Re}(s)\geq 2$, then $\zeta(s)=1+\sum_{n\geq 2}\frac{1}{n^s}$ has

$$\mathrm{Re}(\zeta(s))\geq 1-\sum_{n\geq 2}\frac{1}{n^{\mathrm{Re}(s)}}\geq 1-(\underbrace{\zeta(2)}_{\frac{\pi^2}{6}}-1) > 0.$$

$\Rightarrow$ $\log\zeta(s)$ is well-def. in $\{\mathrm{Re}(s)\geq 2\}$ with

$|\mathrm{Im}\log\zeta(s)|\leq\frac{\pi}{2}$.

$$\to \int_{\mathcal{E}_1}\frac{\zeta'}{\zeta}(s)\,ds = O(1).$$

Now, deal with $\mathcal{E}_2$.

Problem: $\mathcal{E}_2$ can be arbitrarily close to a nontriv. zero, so $\frac{\zeta'}{\zeta}(s)$ can be arbitrarily large.

But the following lemma implies that

$$\mathrm{Im}\left(\int_{\mathcal{E}_2}\frac{\zeta'}{\zeta}(s)\,ds\right)=\sum_{\substack{\rho:\\ |\mathrm{Im}(\rho-s)|\leq 1}}\underbrace{\mathrm{Im}\left(\int_{\mathcal{E}_2}\frac{1}{s-\rho}\,ds\right)}_{\ll 1}+O(\log T)$$

$$\ll \log T.$$

□

Lemma 9.2.5

We have
$$\frac{\zeta'}{\zeta}(s) = \sum_{\substack{\rho : \\ |\mathrm{Im}(\rho - s)| \le 1}} \frac{1}{s-\rho} + O(\log \mathrm{Im}(s))$$
for $\frac{1}{2} \le \mathrm{Re}(s) \le 2$ and large $\mathrm{Im}(s)$.

with only $O(\log T)$ summands

Pf In this region,

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) + O(1) \text{ by Cor 9.2.3.} \qquad (I)$$

$$\| $$

$$\underbrace{\frac{1}{2}\frac{f'}{f}\left(\frac{s}{2}\right)}_{\substack{-\log\pi + \frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) \\ \ll \log|s| \\ \text{by Cor 9.1.3}}} + \underbrace{\frac{\zeta'}{\zeta}(s)}_{\substack{= \sum_{n \ge 1} \frac{\Lambda(n)}{n^s} \ll 1 \\ \text{if } \mathrm{Re}(s) \ge 2}}$$

(with $f(s) = \pi^{-s}\Gamma(s)$ as before)

$\Rightarrow$ For large $t > 0$, (taking $s = 2+it$)

$$\sum_{\rho}\left(\frac{1}{2+it-\rho} + \frac{1}{\rho}\right) \ll \log t$$

$$\Rightarrow \sum_{\rho}\Bigg(\underbrace{\mathrm{Re}\left(\frac{1}{2+it-\rho}\right)}_{\substack{= \frac{\mathrm{Re}(2-\rho)}{|2+it-\rho|^2} \\ \ge \frac{1}{4+|t-\mathrm{Im}(\rho)|^2}}} + \underbrace{\mathrm{Re}\left(\frac{1}{\rho}\right)}_{\ge 0}\Bigg) \ll \log t$$

because $0 \le \mathrm{Re}(\rho) \le 1$.

$$\Rightarrow \#\{\rho : |t - \operatorname{Im}\rho| \le 1\} \ll \log t$$

and

$$\sum_{\substack{\rho: \\ |t-\operatorname{Im}\rho| \ge 1}} \frac{1}{|t-\operatorname{Im}\rho|^2} \ll \log t.$$

Plug this back into $(I)_\rho$ with $t = \operatorname{Im}(s)$:

$$-\frac{\zeta'}{\zeta}(2+it) + \frac{\zeta'}{\zeta}(s) = \sum_\rho \left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\right) + O(1)$$

$$= \underbrace{\sum_{\substack{\rho: \\ |t-\operatorname{Im}\rho| \le 1}} \Big(\frac{1}{s-\rho} - \underbrace{\frac{1}{2+it-\rho}}_{\ll 1}\Big)}_{\ll \log t} + \sum_{\substack{\rho: \\ |t-\operatorname{Im}\rho| > 1}} \underbrace{(\ldots)}_{\frac{2+it-s}{(s-\rho)(2+it-\rho)}} + O(1)$$

Let $t = \mathrm{Im}(s)$ and apply (I) to s and $2+it$:

$$\frac{\zeta'}{\zeta}(s) = \underbrace{\frac{\zeta'}{\zeta}(2+it)}_{\substack{\ll \log t \\ \text{as before}}} + \sum_{\rho}\left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\right) + O(1)$$

$$\underbrace{\sum_{\substack{\rho: \\ |t-\mathrm{Im}(\rho)|\le 1}}}_{\ll \log t \text{ summands}} \left(\frac{1}{s-\rho} - \underbrace{\frac{1}{2+it-\rho}}_{\substack{\ll 1 \\ \text{because} \\ \mathrm{Re}(\rho)\le 1}}\right) = \sum_{\substack{\rho: \\ |\cdot|\le 1}} \frac{1}{s-\rho} + O(\log t)$$

$$\sum_{\substack{\rho: \\ |\cdots|>1}} \left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\right) = \sum_{\substack{\rho: \\ |\cdots|>1}} \frac{2+it-s}{\underbrace{(s-\rho)(2+it-\rho)}_{\ll \frac{1}{|t-\mathrm{Im}(\rho)|^2}}} \ll \log t.$$

$\square$