Thm 7.5 Let χ be a primitive character mod q.

Let $a = \begin{cases} 0, & \chi(-1)=1 \quad (\chi \text{ even}), \\ 1, & \chi(-1)=1 \quad (\chi \text{ odd}). \end{cases}$

Let $\varepsilon(\chi) := \dfrac{\tau(\chi)}{i^a \sqrt{q}} = \dfrac{\tau(\chi)}{\sqrt{\chi(-1)\cdot q}}$.

Let $\xi(s,\chi) := \left(\dfrac{\pi}{q}\right)^{-(s+a)/2} \Gamma\left(\dfrac{s+a}{2}\right) L(s,\chi)$.

Then, $\xi(s,\chi) = \varepsilon(\chi)\cdot \xi(1-s,\overline{\chi})$.

Thm-Rmk 7.6

a) $|\varepsilon(\chi)| = 1$

b) $\varepsilon(\chi)\cdot\varepsilon(\overline{\chi}) = 1$.

c) If χ is real (= real-valued), then $\varepsilon(\chi)=1$.

Pf a), b) easy

c) difficult (see for example Thm 9.15 in Montgomery-Vaughan)

Gauß worked on this for a year...

"Finally, two days ago, I succeeded - not on account of my hard efforts, but by the grace of the Lord. Like a sudden flash of lightning, the riddle was solved. I am unable to say what was the conducting thread that connected what I previously knew with what made my success possible."

Pf of Thm 7.5 for even χ. Let $q > 1$.

Define $\Theta_\chi : \mathbb{R}_{>0} \to \mathbb{R}$ by $\Theta_\chi(u) = \sum_{n \in \mathbb{Z}} \chi(n) e^{-\pi u n^2}$.

Then: a) $\Theta_\chi(u) = O(e^{-u})$ for large u.

b) $\Theta_\chi(u) = \frac{\tau(\chi)}{q} \cdot u^{-1/2} \, \Theta_{\bar\chi}\left(\frac{1}{q^2 u}\right)$

by Lemma 7.1 (twisted Poisson summation) applied to $f(x) = e^{-\pi u x^2}$ with $\hat f(y) = u^{-1/2} e^{-\pi u^{-1} y^2}$

c) $\Theta_\chi(u) = O_q(e^{-u^{-1}})$ for small $u > 0$.

by a), b).

As in the pf of Thm 4.3, $= \int_0^\infty \sum_{n \geq 1} \chi(n) e^{-\pi u n^2} (qu)^{s/2} \frac{du}{u}$

$$\frac{1}{2} \int_0^\infty \Theta_\chi(u) (qu)^{s/2} \frac{du}{u} = \xi(s, \chi) \quad \text{if } \operatorname{Re}(s) > 1.$$

The LHS is holomorphic everywhere, so the eq. holds for all $s \in \mathbb{C}$.

Then, $\xi(s, \chi) = \frac{\tau(\chi)}{\sqrt{q}} \cdot \xi(1-s, \bar\chi)$ follows from b). $\square$

Pf of Thm 7.5 for odd χ.

Note: The previous argument wouldn't work because

$$\Theta_\chi(u) = \sum_{n\in\mathbb{Z}} \chi(n) e^{-\pi u n^2} = \sum_{n\geq 1} \underbrace{(\chi(n)+\chi(-n))}_{0} e^{-\pi u n^2} = 0.$$

Instead, let $\Theta_\chi(u) = \sum_{n\in\mathbb{Z}} \chi(n)\cdot n\, e^{-\pi u n^2}$.

Then: a), c) as before

b) $\Theta_\chi(u) = \dfrac{\tau(\chi)}{i q^2 u^{3/2}} \, \Theta_{\bar\chi}\left(\dfrac{1}{q^2 u}\right)$

by Lemma 7.1 applied to

$$g(x) = x e^{-\pi u x^2} = -\frac{1}{2\pi u} f'(x) \quad (\text{for } f(x) = e^{-\pi u x^2})$$

with $\hat{g}(y) = -\dfrac{1}{2\pi u}\cdot 2\pi i y \cdot \underbrace{\hat{f}(y)}_{u^{-1/2} e^{-\pi u^{-1} y^2}}$.

As in the pf of Thm 4.3,

$$\frac{1}{2}\int_0^\infty \Theta_\chi(u)\,(qu)^{(s+1)/2}\,\frac{du}{u} = \xi(s,\chi) \quad \text{if } \operatorname{Re}(s)>1.$$

$\vdots$

$\square$

Cor 7.7 For primitive characters χ mod q:

a) $L(s,\chi)$ has a simple zero at

$s = 0, -2, -4, \dots$ if χ is even, $q > 1$

$s = -1, -3, -5, \dots$ if χ is odd.

(trivial zeros)

b) All other zeros lie in $\{s \in \mathbb{C} : 0 < \operatorname{Re}(s) < 1\}$.

c) If s is a nontrivial zero of $L(s,\chi)$, then

$1-s$ — $L(s,\overline{\chi})$,

$\overline{s}$ — $L(s,\overline{\chi})$,

$1-\overline{s}$ — $L(s,\chi)$.

Generalized Riemann Hypothesis

For prim. char. χ mod q, the nontriv. zeros of $L(s,\chi)$ satisfy $\operatorname{Re}(s) = \frac{1}{2}$.

Rmk By Thm-Rmk 7.6c), if χ is real, then $L(s,\chi) = L(1-s,\chi)$, which implies that $L(s,\chi)$ can only have a zero of even order at $s = \frac{1}{2}$.

Apparently, it is conjectured that $L(\frac{1}{2},\chi) > 0$, though!

Note (Jonas) We have $L(1,\chi) > 0$. If the GRH holds, then $L(\frac{1}{2},\chi) \geq 0$.

$L(s,\chi) = \prod_p \frac{1}{1-\frac{\chi(p)}{p^s}} > 0$ for $s > 1$.

8. Connection with Algebraic Number Theory

Def The Dedekind zeta function of a number field K is the Dirichlet series

$$\zeta_K(s) = \sum_{\substack{0 \neq \mathfrak{a} \subseteq \mathcal{O}_K \\ \text{ideal}}} \frac{1}{\mathrm{Nm}(\mathfrak{a})^s}$$

$$= \sum_{n \geq 1} \frac{\#\{0 \neq \mathfrak{a} \subseteq \mathcal{O}_K : \mathrm{Nm}(\mathfrak{a}) = n\}}{n^s}$$

$$= \prod_{\substack{\mathfrak{p} \text{ prime} \\ \text{of } K}} \frac{1}{1 - \mathrm{Nm}(\mathfrak{p})^{-s}} .$$

Rmk $\zeta_K(s)$ has a merom. cont. to $\mathbb{C}$ which is hol. except for a simple pole at $s = 1$ with residue $\dfrac{2^{r_1}(2\pi)^{r_2} R_K h_K}{\omega_K \sqrt{|D_K|}}$, (class number formula)

where r_1 = nr. of real emb.
r_2 = nr. of complex emb.
R_K = regulator
h_K = class number
ω_K = nr. of roots of unity
D_K = discriminant.

It satisfies a functional equation. ~~and the~~

Extended Riemann Hypothesis

Every zero of $\zeta_K(s)$ with $0 < \mathrm{Re}(s) < 1$ satisfies $\mathrm{Re}(s) = \frac{1}{2}$.

Rmk $$\zeta_{\mathbb{Q}(\zeta_q)}(s) = \prod_{\substack{\chi \\ \text{char.} \\ \text{mod } q}} L(s,\chi) \cdot \prod_{\substack{\mathfrak{p} \mid q \\ \text{prime} \\ \text{of } K}} \frac{1}{1-Nm(\mathfrak{p})^{-s}} .$$

Rmk If $K \subseteq \mathbb{Q}(\zeta_q)$ is the subfield fixed by

$$H \subseteq (\mathbb{Z}/q\mathbb{Z})^\times = Gal(\mathbb{Q}(\zeta_q)|\mathbb{Q}),$$
$$a \mapsto (\zeta_q \mapsto \zeta_q^a)$$

then $$\zeta_K(s) = \prod_{\substack{\chi \text{ char} \\ \text{mod } q \\ \text{s.t. } \chi(H)=1}} L(s,\chi) \cdot \prod_{\substack{\mathfrak{p} \mid q \\ \text{prime} \\ \text{of } K}} \frac{1}{1-Nm(\mathfrak{p})^{-s}} .$$

Rmk let K be a quadratic numberfield with discriminant D and $q = |D|$ and let χ be the char. mod q given by

$$\chi(p \bmod q) = \begin{cases} 1, & D \text{ quadr. res. mod } p \\ -1, & \text{otherwise} \end{cases}$$

for primes $p \nmid D$. (That this is well-def. uses quadratic reciprocity!)

Then, $$\zeta_K(s) = L(s,\chi_0) L(s,\chi) \cdot \prod_{\mathfrak{p} \mid q} \frac{1}{1-Nm(\mathfrak{p})^{-s}} .$$

__Rmk__ For any finite Galois extension $L|K$ of number fields and any representation ρ of $Gal(L|K)$ over $\mathbb{C}$, one can define an __Artin L-function__

$$L(L|K, \rho, s).$$

__Ex__ $L(\mathbb{Q}(\zeta_q)|\mathbb{Q}, \chi, s) = L(s, \chi)$

where we identify a char. χ mod q with a one-dim. representation $\chi : Gal(\mathbb{Q}(\zeta_q)|\mathbb{Q}) = (\mathbb{Z}/q\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$.

__Artin Conjecture__

If the triv. representation is not a summand of ρ, then $L(L|K, \rho, s)$ has a hol. cont. to $\mathbb{C}$.

9. Zeros

9.1. Hadamard prod. expansion

<u>Def</u> The <u>order</u> of an entire function $f : \mathbb{C} \to \mathbb{C}$ is

$$\inf\{\alpha \geq 0 : f(s) \ll \exp(|s|^\alpha)\} \in [0, \infty].$$

<u>Thm 9.1.1</u> (Hadamard product expansion)

Let f be an entire function of order 1 with $f(0) \neq 0$.

Then, there exist $A, B \in \mathbb{C}$ such that

$$f(s) = e^{A+Bs} \prod_{\substack{\rho \text{ root} \\ \text{of } f \\ \text{(with} \\ \text{mult.)}}} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \qquad \text{for all } s \in \mathbb{C},$$

where the product is locally uniformly convergent.

Also,

$$\frac{f'}{f}(s) = B + \sum_{\rho \cdots} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right),$$

where the sum is locally uniformly absolutely convergent.

<u>Warning</u> $\sum_\rho \frac{1}{\rho}$ might not converge!