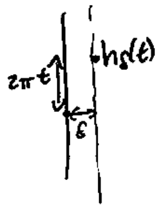
Reminder: $a_1, a_2, \dots \geq 0$

$$f(x) = \sum_{n \leq x} a_n$$

$$g(u) = \frac{f(e^u)}{e^u}$$

Goal: $g(u) \xrightarrow{u \to \infty} 1$

$H(s)$ cont. on $\{\mathrm{Re}(s) \geq 1\}$



$$h_\delta(t) = H(1 + \delta + 2\pi i t)$$

$$\phi_\delta(u) = \begin{cases} (g(u)-1)e^{-u\delta} & , u \geq 0 \\ 0 & , u < 0 \end{cases}$$

$$h_\delta = \widehat{\phi_\delta}$$

Fake pf:

$$\begin{array}{ccc} \phi_\delta(u) & = & \hat{h}_\delta(-u) \\ \downarrow & \delta \to 0 & \downarrow \\ g(u) - 1 & = & \hat{h}_0(-u) \\ & & \downarrow \\ & & 0 \end{array}$$

- Idea: Smoothen ϕ_δ by taking its convolution with a Fejér kernel K_λ (or a kernel from Problem 1a on Set 4)
  $\phi_\delta * K_\lambda \in L^1(\mathbb{R})$ because $\phi_\delta, K_\lambda \in L^1(\mathbb{R})$.

- $\widehat{\phi_\delta * K_\lambda} = \widehat{\phi_\delta} \cdot \widehat{K_\lambda} = h_\delta \hat{k}_\lambda \in L^1(\mathbb{R})$
  because $\widehat{K_\lambda}$ is compactly supported and h_δ is continuous.

$$\Rightarrow (\phi_\delta * K_\lambda)(v) = \widehat{h_\delta \cdot \widehat{K_\lambda}}\,(-v)$$

Thm 2.2.3

$$\Rightarrow \int_{\mathbb{R}} \phi_\delta(v-u)\, K_\lambda(u)\, du = \int_{\mathbb{R}} h_\delta(t) \underbrace{\widehat{K_\lambda}(t)}_{\text{cpt. support}} e^{2\pi i t v}\, dt$$

monotone convergence $\downarrow$ $\qquad \delta \to 0 \qquad$ $\downarrow$ a) ($h_\delta \to h_0$ locally uniformly)

$$\int_{\mathbb{R}} \phi_0(v-u)\, K_\lambda(u)\, du = \int_{\mathbb{R}} h_0(t)\, \widehat{K_\lambda}(t)\, e^{2\pi i t v}\, dt$$

$$\| $$

$$\widehat{\underbrace{h_0}_{\text{cont.}} \cdot \underbrace{\widehat{K_\lambda}}_{\text{cpt. supp.}}}\,(-v) \xrightarrow[\text{Thm 2.2.1: Riemann-Lebesgue}]{v \to \infty} 0$$

$\Rightarrow \in L^1$

$$\Rightarrow \int_{\mathbb{R}} \underbrace{\phi_0(v-u)}_{\begin{cases} g(v-u)-1, & u \le v, \\ 0, & u > v, \end{cases}} K_\lambda(u)\, du \xrightarrow{v \to \infty} 0 \quad \text{for any } \lambda > 0. \quad (I)$$

alone: $\text{LHS} \xrightarrow{\lambda \to \infty} \phi_0(v) = g(v) - 1.$ "uniformly"

~~Note: this would be automatic if ϕ_0 were continuous.~~

Note: $f(x) = \sum_{n \le x} a_n$ is increasing, so $g(u) = \frac{f(e^u)}{e^u}$ satisfies

$$g(v+u) \ge g(v)\, e^{-u} \quad \text{for any } u \ge 0.$$

Let $r_\lambda(v) := \int_{-\infty}^{v} g(v-u)\, K_\lambda(u)\, du = \int_{\mathbb{R}} \underbrace{g(v-u)\, K_\lambda(u)}_{\ge 0}\, du.$

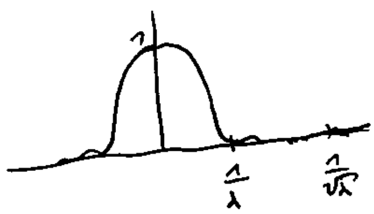
$$(I) \Leftrightarrow r_\lambda(v) \xrightarrow{v \to \infty} \int_{\mathbb{R}} K_\lambda(u)\, du = \widehat{K_\lambda}(0) = 1.$$

$$\Rightarrow r_\lambda(v) \geq \int_{-\frac{1}{\sqrt{\lambda}}}^{\frac{1}{\sqrt{\lambda}}} g(v-u)\, K_\lambda(u)\, du$$

$$= g\left(v-\tfrac{1}{\sqrt{\lambda}}\right) \underbrace{e^{-2/\sqrt{\lambda}}} \int_{-\frac{1}{\sqrt{\lambda}}}^{\frac{1}{\sqrt{\lambda}}} K_\lambda(u)\, du$$

$$\downarrow \quad \lambda\to\infty \quad \downarrow$$

$$1$$



Let $\varepsilon>0$. Pick λ large enough so that

$$e^{-2/\sqrt{\lambda}} \int K_\lambda(u)\,du \geq \frac{1}{1+\varepsilon}$$

$$\Rightarrow (1+\varepsilon)\, r_\lambda(v) \geq g\left(v-\tfrac{1}{\sqrt{\lambda}}\right) \qquad \forall v$$

$$\downarrow v\to\infty$$

$$1$$

$$\Rightarrow \limsup_{v\to\infty} g(v) \leq 1+\varepsilon \qquad \forall \varepsilon>0$$

$$\Rightarrow \quad \text{—"—} \quad \leq 1.$$

In particular, $g(v) << 1$.

$$\Rightarrow r_\lambda(v) \leq \int_{-\frac{1}{\sqrt{\lambda}}}^{\frac{1}{\sqrt{\lambda}}} g\left(v + \frac{1}{\sqrt{\lambda}}\right) \underset{\substack{\downarrow \\ 1}}{e^{2/\sqrt{\lambda}}} \, {\scriptstyle \lambda \to \infty} \, \underset{\substack{\downarrow \\ 1}}{K_\lambda(u)\,du}$$

$$+ \underset{\substack{\downarrow \lambda \to \infty \\ 0}}{\mathcal{O}\left(\int_{\mathbb{R} \setminus [-\frac{1}{\sqrt{\lambda}}, \frac{1}{\sqrt{\lambda}}]} K_\lambda(u)\,du \right)}$$

$$\underset{\substack{\uparrow \\ \text{as before}}}{\Longrightarrow} \liminf_{v \to \infty} g(v) \geq 1.$$

$$\Rightarrow \lim_{v \to \infty} g(v) = 1.$$

$\square$

6. Dirichlet L-series

Def Let $q \geq 1$. A <u>(multiplicative) charachter</u> mod q is a group hom. $\chi : (\mathbb{Z}/q\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$.

Ex The <u>trivial character</u> χ_0 with $\chi_0(x) = 1 \ \forall x \in (\mathbb{Z}/q\mathbb{Z})^\times$.

Ex If $q \neq 2$ is prime:

$$\chi_0(x) = \begin{cases} 1, & x \text{ quadr. res. mod } q, \\ -1, & \text{otherwise}. \end{cases}$$

(Note: $\chi_0(x) \equiv x^{\frac{q-1}{2}} \bmod q$.)

Rmk Each $\chi(x)$ is a (primitive) r-th root of unity for some $r \mid \varphi(q)$.

Pf
$\chi(x)^{\varphi(q)} = \chi(x^{\varphi(q)}) = \chi(1) = 1.$ $\square$

Rmk The finite (abelian) group $(\mathbb{Z}/q\mathbb{Z})^\times$ is isomorphic to a product of cyclic groups: $(\mathbb{Z}/q\mathbb{Z})^\times \cong \mathbb{Z}/k_1\mathbb{Z} \times \dots \times \mathbb{Z}/k_r\mathbb{Z}$.
The group homomorphisms

$$\mathbb{Z}/k_1\mathbb{Z} \times \dots \times \mathbb{Z}/k_r\mathbb{Z} \longrightarrow \mathbb{C}^\times$$

are the maps

$$(a_1, \dots, a_r) \longmapsto \zeta_{k_1}^{a_1 i_1} \cdots \zeta_{k_r}^{a_r i_r}$$

for $(i_1, \dots, i_r) \in \mathbb{Z}/k_1\mathbb{Z} \times \dots \times \mathbb{Z}/k_r\mathbb{Z}$.
In particular, $\#\{\chi\} = \#(\mathbb{Z}/q\mathbb{Z})^\times = \varphi(q)$.

Lemma 6.1

a) For any χ,
$$\sum_{x \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(x) = \begin{cases} \varphi(q), & \chi = \chi_0, \\ 0, & \chi \neq \chi_0. \end{cases}$$

b) For any x,
$$\sum_{\chi} \chi(x) = \begin{cases} \varphi(q), & x = 0 \bmod q, \\ 0, & x \neq 0 \bmod q. \end{cases}$$

Pf HW. $\square$

Def The Dirichlet L-series for χ is

$$L(s,\chi) := \sum_{\substack{n \geq 1: \\ \gcd(n,q)=1}} \frac{\chi(n \bmod q)}{n^s}.$$

Rmk Often, people extend χ to $\mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$ by letting $\chi(x)=0$ if $x \notin (\mathbb{Z}/q\mathbb{Z})^\times$. Then $L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n \bmod q)}{n^s}$.

Note that the corr. function $\begin{array}{l} \mathbb{N} \to \mathbb{C} \\ n \mapsto \chi(n \bmod q) \end{array}$ is completely multiplicative.

Rmk Formally, $L(s,\chi) = \prod_{p \nmid q} \frac{1}{1-\frac{\chi(p)}{p^s}}$.

Ex $L(s,\chi_0) = \prod_{p\nmid q} \frac{1}{1-\frac{1}{p^s}} = \zeta(s) \cdot \prod_{p \mid q} \left(1 - \frac{1}{p^s}\right)$,

which is holomorphic except for a simple pole at $s=1$ with residue $\prod_{p\mid q}\left(1-\frac{1}{p}\right) = \frac{\varphi(q)}{q}$.

Lemma 6.2 If $\chi \neq \chi_0$, then $L(s,\chi)$ has a holomorphic continuation to $\mathbb{C}$.

Pf χ is periodic, $\sum_x \chi(x) = 0$. Apply Thm 3.2.3. $\square$

Thm 6.3 $L(s,\chi)$ has no zeros with $\mathrm{Re}(s) \ge 1$.

Pf We have

$$-\frac{L'(s,\chi)}{L(s,\chi)} = \sum_{n=p^k} \frac{\chi(n)\log p}{n^s}.$$

Let $f(s) := \prod_{\chi} L(s,\chi)$.

$$\Rightarrow -\frac{f'(s)}{f(s)} = \sum_{\chi}\left(-\frac{L'(s,\chi)}{L(s,\chi)}\right) = \sum_{n=p^k} \frac{\sum_{\chi}\chi(n)\log p}{n^s}$$

$$\underset{\text{Lemma 6.1b}}{=} \sum_{\substack{n=p^k:\\ n\equiv 1 \bmod q}} \frac{\varphi(q)\log p}{n^s}.$$

This Dirichlet series has nonnegative coefficients and satisfies

$$\varphi(q)\sum_{\substack{m=p^k:\\ \gcd(m,q)=1}} \frac{\log p}{m^{\varphi(q)s}} \overset{n=m^{\varphi(q)}\equiv 1 \bmod q}{\le} -\frac{f'(s)}{f(s)} \le \varphi(q)\sum_{n=p^k}\frac{\log p}{n^s} = \varphi(q)\cdot\left(-\frac{\zeta'(s)}{\zeta(s)}\right)$$

for $s \in \mathbb{R}$.

$$\| $$

$$\varphi(q)\cdot\left(-\frac{\zeta'(\varphi(q)s)}{\zeta(\varphi(q)s)}\right) + O_q(1)$$

It therefore has abscissa of convergence $\frac{1}{\varphi(q)} \le \sigma_c \le 1$, so must have a pole at σ_c by Thm 3.2.2.

Since the coeff. are ≥ 0 it must be a pole of positive residue. (and therefore $\lim\limits_{s \to \sigma_c^+} \left(-\frac{f'(s)}{f(s)}\right) = \infty$)

$\Rightarrow f(s) = \prod\limits_{\chi} L(s,\chi)$ has a pole.

The only pole of any factor is a simple pole at $s=1$ for $\chi = \chi_0$.

$\Rightarrow f(s)$ has a simple pole at $s=1$, and is holomorphic everywhere else, and $L(1,\chi) \neq 0 \; \forall \chi$.

$\Rightarrow$ By e.g. Problem 4b on Pset 4 (or the same proof as in Thm 4.5), $f(s)$ has no zeros with $\mathrm{Re}(s) \geq 1$.

$\Rightarrow L(s,\chi) = 0$ for $\mathrm{Re}(s) \geq 1$. $\square$