5. The Wiener-Ikehara Theorem

5.1. Statement

Thm 5.1.1 (Wiener-Ikehara)

Let $a_1, a_2, \ldots \geq 0$ and $d > 0$ and assume that $D(a,s)$ can be meromorphically continued to (a neighborhood of) $\{s \in \mathbb{C} : \mathrm{Re}(s) \geq d\}$, holomorphic except for a simple pole at $s = d$ with $\lim\limits_{s \to d} D(a,s) \cdot (s-d) = A$.

Then, $\sum\limits_{n \leq x} a_n \sim \frac{A}{d} \cdot x^d$ for $x \to \infty$.

__Ex__ $D(\mathbb{1}, s) = \zeta(s) \rightsquigarrow d = 1,\ A = 1$

$$\sum_{n \leq x} 1 \sim x$$

__Ex__ $D(id^k, s) = \zeta(s-k)$ with $k > -1$

$\rightsquigarrow d = k+1,\ A = 1$

$$\sum_{n \leq x} n^k \sim \int_1^x t^k \, dt \sim \frac{1}{k+1} x^{k+1}.$$

__Ex__ $D(\mathbb{1}_{square}, s) = \zeta(2s) \rightsquigarrow d = \frac{1}{2},\ A = \frac{1}{2}$

$$\sum_{\substack{n \leq x \\ square}} 1 = \sum_{m \leq x^{1/2}} 1 \sim x^{1/2}$$

$\underline{\text{Ex}}$ $D(\sigma, s) = \zeta(s-1)\zeta(s) \rightsquigarrow d=2, \ A = \zeta(2)$

σ: sum of divisors

(poles at $s=1,2$)

$$\Rightarrow \sum_{n \le x} \sigma_n \sim \frac{\zeta(2)}{2} \cdot x^2$$

$\underline{\text{Ex}}$ $a_n = \#\{(c,d) : c,d \ge 1, \ n = c^2 d\}$

$D(a,s) = \zeta(2s)\zeta(s) \rightsquigarrow d = 1, \ A = \zeta(2)$

$$\Rightarrow \sum_{\substack{c,d: \\ c^2 d \le x}} 1 \sim \zeta(2) \cdot x$$

$\underline{\text{Ex}}$ $D(\Lambda, s) = -\frac{\zeta'(s)}{\zeta(s)} \rightsquigarrow d = 1, \ A = 1$

(pole at $s=1$, no other poles with $\mathrm{Re}(s) \ge 1$)

$$\Rightarrow \sum_{n \le x} \Lambda(n) \sim x$$

$$\sum_{\substack{p,e:\, p^e \le x \\ (p \text{ prime},\, e \ge 1)}} \log p = \sum_{p \text{ prime}} \log p + O\Big(\underbrace{\sum_{\substack{e \ge 2 \\ \le \log_2 x}} \underbrace{\sum_{\substack{m \ge 2: \\ m^e \le x}} \log n}_{O(x^{1/2} \log n)}}_{O(x^{1/2} (\log n)^2)}\Big)$$

$$\Rightarrow \sum_{p \text{ prime}} \log p \sim x$$

$\Leftrightarrow$ PNT

Problem 3 on Pset 3

Thm 5.1.2 (Kato: A remark on the Wiener-Ikehara Tauberian Theorem)

Let $a_1, a_2, \ldots \geq 0$ and $c, m \geq 1$ and $d > 0$ and assume that $D(a,s)^m$ can be meromorphically continued to (a nbhd of) $\{s \in \mathbb{C} : \Re(s) \geq d\}$, holomorphic except for a pole of order c at $s = d$ with $\lim_{s \to d} D(a,s)^m \cdot (s-d)^c = A^m$. $(A > 0)$

("pole of order c/m")

Then, $\sum_{n \leq X} a_n \sim \frac{A}{d \cdot \Gamma(\frac{c}{m})} \cdot X^d (\log X)^{\frac{c}{m} - 1}$.

Ex $D(d,s) = \zeta(s)^2 \rightsquigarrow d = 1, \ \frac{c}{m} = \frac{2}{1}, \ A = 1.$

$\sum_{n \leq X} d_n \sim X \log X$

Ex $D(d^{(3)}, s) = \zeta(s)^3 \rightsquigarrow d = 1, \ \frac{c}{m} = \frac{3}{1}, \ A = 1$

$\sum d_n^{(3)} \sim \frac{1}{2} X (\log X)^2$

[Ex with $m > 1$: later...]

5.2. Proof

We'll now prove Thm 5.1.1 following chapter 3.3 in Murty.

Rmk It suffices to prove Thm 5.1.1 for $d=1$.

Pf Consider the sequence $b_n = a_n \cdot n^{1-d}$.

$D(b,s) = D(a, s+d-1)$ has merom. cont. with pole at $s=1$. $\lim_{s\to 1} D(b,s)\cdot(s-1) = A$.

$$\underset{\text{d=1 case}}{\Rightarrow} \quad \sum_{n\le x} b_n \sim A\cdot x$$

Apply Abel summation to estimate

$$\sum_{n\le x} a_n = \sum_{n\le x} b_n \cdot n^{d-1}. \qquad \square$$

Lemma 5.2.1 Let $a_1, a_2, \dots \ge 0$ and assume that $D(a,s)$ has abscissa of convergence $\sigma_c > 0$.

Then, $\sum_{n\le x} a_n \ll_q x^q$ for all $q > \sigma_c$ and $x \ge 1$.

Pf $$\sum_{n\le x} a_n \le \sum_{n=1}^{\infty} a_n \cdot \left(\frac{x}{n}\right)^q = x^q \cdot \underbrace{D(a,q)}_{<\infty}. \qquad \square$$

Pf of Thm 5.1.1

W.l.o.g. $d=1$, $A=1$.

Let $f(x) = \sum_{n \le x} a_n$.

Abel summation for $f(x)$, $\frac{1}{x^s}$ shows for $\mathrm{Re}(s) > 1$:

$$F(s) := D(a,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \underset{\uparrow}{=} s \cdot \int_1^{\infty} \frac{f(x)}{x^{s+1}}\, dx$$

$\frac{f(x)}{x^s} \to 0$ by Lemma 5.2.1

$$\underset{\substack{\uparrow \\ x = e^u}}{=} s \cdot \int_0^{\infty} f(e^u) e^{-us}\, du$$

$$\Rightarrow H(s) := \frac{F(s)}{s} - \frac{1}{s-1} = \int_0^{\infty} \left(f(e^u)e^{-u} - 1\right) e^{-u(s-1)}\, du$$

$$= \int_0^{\infty} (g(u) - 1) e^{-u(s-1)}\, du \quad \text{for } \mathrm{Re}(s) > 1.$$

with $g(u) := f(e^u)e^{-u} = \dfrac{\sum_{n \le e^u} a_n}{e^u}$.

Goal: $g(u) \xrightarrow{u \to \infty} 0$.

By assumption, $H(s)$ can be holomorphically continued to $\{ s \in \mathbb{C} \mid \mathrm{Re}(s) \ge 1 \}$.

For any $\delta \geq 0$ and $t \in \mathbb{R}$, let

$$h_\delta(t) = H(1+\delta+2\pi i t).$$

Let $\phi_\delta(u) := \begin{cases} (g(u)-1)e^{-u\delta}, & u \geq 0, \\ 0, & u < 0. \end{cases}$

$$\Rightarrow h_\delta(t) = \int_0^\infty (g(u)-1)e^{-u\delta}\, e^{-2\pi i u t}\, du$$

$$= \widehat{\phi_\delta}(t).$$

~~Note: $|(g(u)-1)e^{-u\delta}| \leq C_\delta e^{-u\delta/2}$ by Lemma 5.2.1.~~

<u>Fake proof:</u> $h_\delta(t) = \widehat{\phi_\delta}(t) \quad \forall t$

$$\Rightarrow \phi_\delta(u) = \widehat{h_\delta}(-u) \quad \forall u$$

$$\begin{tikzcd} \phi_\delta(u) \arrow[d] \arrow[r, phantom, "="] & \widehat{h_\delta}(-u) \arrow[d] \arrow[l, phantom, "\delta \to 0"'] & \\ g(u)-1 \arrow[r, phantom, "="] & \widehat{h_0}(-u) \arrow[r, "u \to \infty"] & 0 \end{tikzcd}$$

(Thm 2.2.1: Riemann-Lebesgue Lemma)

Issues: • h_δ converges to h_0 pointwise, but perhaps not uniformly.

Maybe $\widehat{h_\delta}$ doesn't even converge to $\widehat{h_0}$ pointwise.

• Maybe $\hat{h}_0$ doesn't even lie in $L^1(\mathbb{R})$.

Maybe $\hat{h}_0$ —"— .

Note: a) At least h_δ converges to h_0 locally uniformly (because H is continuous).

b) We have $\phi_\delta(v) \ll_\delta e^{-v\delta/2}$ by Lemma 5.2.1, so

in particular $\phi_\delta \in L^1(\mathbb{R})$ and $L^2(\mathbb{R})$.

$$\Rightarrow \quad h_\delta = \widehat{\phi_\delta} \in L^2(\mathbb{R}).$$

Solution: Use the Fejér kernel

$$K(x) = \left(\frac{\sin(\pi x)}{\pi x}\right)^2 \geq 0 \qquad (\, K(0) = 1\,)$$

with

$$\hat{K}(t) = \begin{cases} 1 - |t| & , \ |t| \leq 1, \\ 0 & , \ |t| \geq 0. \end{cases} \qquad \text{(compactly supported, } \geq 0\text{)}$$

Let $K_\lambda(x) = \lambda \cdot K(\lambda x)$. $\Rightarrow \widehat{K_\lambda}(t) = \hat{K}\left(\frac{t}{\lambda}\right)$.