The assumption that $a_1, a_2, \ldots \geq 0$ is necessary:

Thm 3.2.3 Let the sequence $a_1, a_2, \ldots$ be periodic with period m and assume $a_1 + \ldots + a_m = 0$.

Then, $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ (with $\sigma_c \leq \sigma_a \leq 1$ because $a_n = O(1)$) has a holomorphic continuation to $\mathbb{C}$.

Ex $\sum \frac{(-1)^{n-1}}{n^s}$, $\sum \exp(2\pi i n/m) \cdot \frac{1}{n^s}$, ...

Pf Apply Abel summation to $\sum_{n \leq x} a_n$ and $\frac{1}{t^s}$:

$\| \leftarrow$ periodic (because $a_1 + \ldots + a_m = 0$)

$O(1)$

For $\mathrm{Re}(s) > 1$:

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \Big[\underbrace{\sum_{n \leq t} a_n}_{O(1)} \cdot \frac{1}{t^s} \Big]_{t=1}^{\infty} - \int_1^{\infty} \underbrace{\sum_{n \leq t} a_n}_{O(1)} \cdot \frac{-s}{t^{s+1}} \, dt$$

The RHS is a hol. cont. to $\{ s \in \mathbb{C} : \mathrm{Re}(s) > 0 \}$.

Keep integrating by parts as in the construction of the Euler-Maclaurin formulas, making sure to keep the first function bounded ...

(previously the Bernoulli fcts)

$\square$

4. The functional equation

Def The *theta function* $\Theta : \mathbb{R}_{>0} \to \mathbb{R}$ is given by

$$\Theta(v) = \sum_{n \in \mathbb{Z}} e^{-\pi v n^2}.$$

Thm 4.1

a) $\Theta(v) - 1 = \mathcal{O}(e^{-v})$ for large v.

b) $\Theta(v^{-1}) = v^{1/2}\,\Theta(v) \quad \forall v > 0$

Pf a) easy
b) Poisson summation (Problem 1 on Pset 2)

$\square$

Def The *gamma function* Γ is the meromorphic continuation of the function given by $\Gamma(s) = \int_0^\infty x^{s} e^{-x} \frac{dx}{x}$ for $\mathrm{Re}(s) > 0$.

Thm 4.2

a) $s\Gamma(s) = \Gamma(s+1) \quad \forall s \in \mathbb{C}$

b) $\Gamma(n+1) = n! \quad \forall n \geq 0.$

c) $\Gamma(s)$ has simple poles at $s = 0, -1, -2, \ldots$ and no other poles.

d) $\Gamma(s)$ has no zeros.

e) $\Gamma(s)\Gamma(1-s) = \dfrac{\pi}{\sin(\pi s)}$

f) $\log \Gamma(s) = s \log s - s - \frac{1}{2}\log s + C + \mathcal{O}_\varepsilon(|s|^{-1})$ if $\arg(s) \in [-\pi+\varepsilon, \pi-\varepsilon]$
(Stirling's approximation) with $C = \frac{1}{2}\log(2\pi)$.

Def The *xi function* is

$$\xi(s) = \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s).$$

Rmk There are other conventions as well!

Thm 4.3 (Functional equation)

We have $\xi(s) = \xi(1-s)$.

Pf First, note that for $\mathrm{Re}(s)>0$:

$$\int_0^\infty e^{-\pi u n^2}\, u^{s/2}\,\frac{du}{u} = \pi^{-s/2}\cdot\frac{1}{n^s}\cdot\underbrace{\int_0^\infty x^{s/2}\,e^{-x}\,\frac{dx}{x}}_{\Gamma(s/2)}.$$

$x = \pi u n^2$

$\Rightarrow$ For $\mathrm{Re}(s) > 1$:

$$\frac12\int_0^\infty \underbrace{(\Theta(u)-1)}_{\substack{\sum_{n\neq 0} e^{-\pi u n^2}\\ =\frac12\sum_{n\geq 1} e^{-\pi u n^2}}}\, u^{s/2}\,\frac{du}{u} = \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s) = \xi(s).$$

$$\Rightarrow 2\xi(s) = \int_1^\infty u^{s/2}(\Theta(u)-1)\frac{du}{u} + \int_0^1 u^{s/2}(\Theta(u)-1)\frac{du}{u}$$

$$\underset{u=t^{-1}}{=} \quad '' \quad + \int_1^\infty t^{-s/2}\left(t^{1/2}\Theta(t)-1\right)\frac{dt}{t}$$

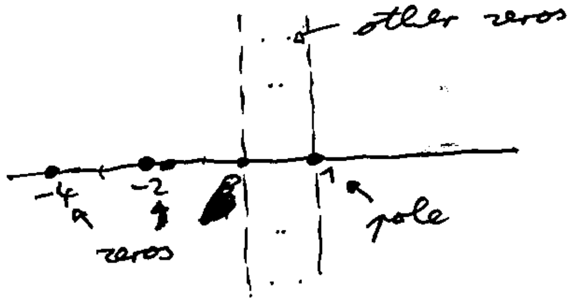
$$= \quad '' \quad + \int_1^\infty t^{(1-s)/2}(\Theta(t)-1)\frac{dt}{t} + \int_1^\infty \left(t^{(1-s)/2} - t^{-s/2}\right)\frac{dt}{t}$$

$$= \quad '' \quad + \quad '' \quad - \frac{2}{1-s} - \frac{2}{s}$$

According to Thm 4.1 a), the RHS is meromorphic for all s, so the equation holds for all s.
The RHS is unchanged when replacing s by $1-s$. □

Cor 4.4

a) $\zeta(s)$ has a simple zero at $s = -2, -4, \ldots$ (trivial zeros)
b) All other (nontrivial) zeros lie in $\{s \in \mathbb{C} : 0 \leq \operatorname{Re}(s) \leq 1\}$.
c) If s is a nontrivial zero, then so are $1-s$, $\bar{s}$, $1-\bar{s}$.



Riemann Hypothesis All nontrivial zeros satisfy $\operatorname{Re}(s) = \frac{1}{2}$.

Pf a) $\xi(s) = \pi^{-s/2} \underbrace{\Gamma(s/2)}_{\text{simple pole}} \zeta(s) = \xi(1-s) = \pi^{-(1-s)/2} \underbrace{\Gamma((1-s)/2)}_{\text{neither zero nor pole}} \underbrace{\zeta(1-s)}_{\text{neither zero nor pole}}$

(cf. Cor 3.1.6)

at $s = -2, -4, \ldots$

b) $\operatorname{Re}(s) \leq 1$ by Cor 3.1.6.
If $\operatorname{Re}(s) < 0$, then $\operatorname{Re}(1-s) > 1$, so RHS has no zero. Also, $\Gamma(s/2)$ has no pole unless $s = -2, -4, \ldots$

c) $\zeta(s) = 0$, $\Gamma(s/2), \Gamma((1-s)/2)$ no zeros/poles $\Rightarrow \zeta(1-s) = 0$.
$\zeta(\bar{s}) = \overline{\zeta(s)} = 0$. □

Thm 4.5 $\zeta(s)$ has no zeros with $\mathrm{Re}(s)=1$.

Pf By Problem 2c on Pset 3, we have

$$-\frac{\zeta'}{\zeta}(s)=\sum_{n\geq 1}\frac{\Lambda(n)}{n^s}$$

with $\Lambda(n)=\begin{cases}\log p, & n=p^e\ (e\geq 1),\\ 0, & \text{otherwise.}\end{cases}$

Clearly, $0\leq \Lambda(n)\ll_\varepsilon n^\varepsilon$.

$\Rightarrow D(\Lambda,s)$ has $\sigma_c\leq 1$.

If $\zeta(s)=f(s)\cdot(s-s_0)^k$ with $f(s)$ holomorphic and nonzero at $s=s_0$

(meaning: $\zeta(s)$ has zero of order k or pole of order $-k$ at $s=s_0$), then

$-\frac{\zeta'}{\zeta}(s)$ has a simple pole at $s=s_0$ with residue $-k$.

Note: Since $-\frac{\zeta'}{\zeta}(s)$ is holomorphic in $\{s\in\mathbb{C}:\mathrm{Re}(s)>1\}$, this again proves that $\zeta(s)$ has no zeros with $\mathrm{Re}(s)>1$.

Now, observe that

$$3+4\cos\theta+\cos 2\theta = 2(1+\cos\theta)^2 \geq 0$$

for all $\theta \in \mathbb{R}$.

$$\Rightarrow 3+4\,\mathrm{Re}\left(\frac{1}{n^{it}}\right)+\mathrm{Re}\left(\frac{1}{n^{2it}}\right) \geq 0 \quad \forall t \in \mathbb{R}.$$

$$\Rightarrow 3\cdot\underbrace{\sum \frac{\Lambda(n)}{n^{\sigma}}}_{-\frac{\zeta'}{\zeta}(\sigma)} + 4\,\mathrm{Re}\Big(\underbrace{\sum \frac{\Lambda(n)}{n^{\sigma+it}}}_{-\frac{\zeta'}{\zeta}(\sigma+it)}\Big) + \mathrm{Re}\Big(\underbrace{\sum \frac{\Lambda(n)}{n^{\sigma+2it}}}_{-\frac{\zeta'}{\zeta}(\sigma+2it)}\Big) \geq 0 \qquad (\mathrm{I})$$

$\forall \sigma > 1,\ t \in \mathbb{R}$

Fact. Assume ζ has a zero of order $k \geq 0$ at $1+it$ and of ~~order~~ $l \geq 0$ at $1+2it$.

$$\Rightarrow -\frac{\zeta'}{\zeta}(\sigma) = \frac{1}{\sigma-1}+O_t(1) \quad \text{for } \sigma \to 1,$$

$$\ldots(\sigma+it) = -\frac{k}{\sigma-1}+O_t(1) \quad \text{—"—},$$

$$\ldots(\sigma+2it) = -\frac{l}{\sigma-1}+O_t(1) \quad \text{—"—}.$$

$$\overset{(\mathrm{I})}{\Rightarrow} 3-4k-l \geq 0 \quad \Rightarrow 4k \leq 3 \Rightarrow k=0.$$

$\Rightarrow \zeta$ has no zero at $1+it$. $\square$