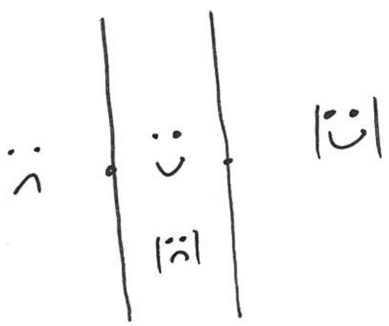
Rmk/Def Similarly, there is a number $\sigma_a = \sigma_a(a) = \sigma_c(|a|) \in \mathbb{R}\cup\{\pm\infty\}$, called the <u>abscissa of absolute convergence</u> such that $\sum \left|\frac{a_n}{n^s}\right|$ converges if $\mathrm{Re}(s) > \sigma_a$
$= \sum \frac{|a_n|}{n^{\mathrm{Re}(s)}}$

and doesn't converge if $\mathrm{Re}(s) < \sigma_a$.

Lemma 3.1.3 $\quad \sigma_c \le \sigma_a \le \sigma_c + 1.$



Rmk This is unlike for power series, where radius of conv. = radius of abs. conv.

$\sigma_c \quad \sigma_a$

Pf Let $s_1, s_2 \in \mathbb{C}$, $\mathrm{Re}(s_1) + 1 < \mathrm{Re}(s_2)$.

$$\sum \frac{a_n}{n^{s_1}} \text{ conv.} \Rightarrow \frac{a_n}{n^{s_1}} = O(1) \Rightarrow \frac{a_n}{n^{s_2}} = O\left(\frac{1}{n^{s_2 - s_1}}\right)$$

$$\Rightarrow \sum \left|\frac{a_n}{n^{s_2}}\right| \text{ conv.}$$

$\square$

Ex $\sum \frac{(-1)^{n-1}}{n^s} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \pm \dots$ has $\sigma_c = 0$, $\sigma_a = 1$.

(or more generally $\sum \frac{e^{2\pi i n/m}}{n^s}$ for $\underset{\in \mathbb{Z}}{m} \ge 2$)

(or even more generally $\sum \frac{a_n}{n^s}$ with $a_1, a_2, \dots$ periodic, not all 0, and $\sum_{n \le x} a_n$ bounded)

~~Prop. If $\mathrm{Re}(s) > \sigma_c^{(a)}$, then~~

$$\cancel{\frac{d}{ds} \sum_{n\geq 1} \frac{a_n}{n^s} = \underbrace{\sum_{n\geq 1} \frac{-a_n \log n}{n^s}}_{\text{convergent}}}$$

Prop. If $\sum \frac{a_n}{n^s} = 0$ for all s with $\mathrm{Re}(s) > \underset{\mathbb{R}}{\overset{\Uparrow}{\sigma}}$,

then $a_n = 0$ for all n.

Pf. Assume a_m is the first nonzero entry.

$$\frac{a_m}{\cancel{m^s}} = - \sum_{n>m} \underbrace{\frac{a_n}{(n/m)^s}}_{\xrightarrow{s\to\infty} 0} \qquad \text{for suff. large } \mathrm{Re}(s)$$

$$\Rightarrow |a_m| \leq \underbrace{\sum_{n>m} \frac{|a_n|}{|n/m|^s}}_{\substack{\text{mon.}\\ \text{decreasing,}\\ \to 0\\ \text{for } s\to\infty}} \qquad \text{for suff. large } s \in \mathbb{R}$$

$\mathbb{R}$ converges for $s > \sigma + 1$

$$\xrightarrow{s\to\infty} 0$$

$\Rightarrow a_m = 0.$ $\square$

Lemma 3.1.4 If $D(a,s)$ and $D(b,s)$ are absolutely convergent, then $D(a*b,s)$ is, and $D(a*b,s) = D(a,s)D(b,s)$.

Pf Just rearrange summands. $\square$

Lemma 3.1.5 If a is multiplicative and $D(^*a,s)$ converges absolutely, then: a) the product

$$\prod_p \sum_{k\geq 0} \frac{a_{p^k}}{p^{ks}} \text{ converges to } D(a,s).$$

b) If $D(a,s)=0$, then at least one factor $\sum_{k\geq 0} \frac{a_{p^k}}{p^{ks}}$ is 0.

Pf a) $$\prod_{p\leq P} \sum_{k\geq 0} \frac{a_{p^k}}{p^{ks}} = \sum_{\substack{n\geq 1 \\ \text{not divisible} \\ \text{by any } p>P}} \frac{a_n}{n^s} \xrightarrow[P\to\infty]{} D(a,s).$$

b) If $D(a,s)=0$

$$\overset{\shortparallel}{\prod_p \sum_k \frac{a_{p^k}}{p^{ks}}}$$

and $\sum_{k\geq 0} \frac{a_{p^k}}{p^{ks}} \neq 0$ for all p, then

$$\prod_{p>P} \sum_{k\geq 0} \frac{a_{p^k}}{p^{ks}} = 0 \text{ for all } P.$$

$$\shortparallel$$

$$\sum_{\substack{n\geq 1 \\ \text{only divisible} \\ \text{by } p>P}} \frac{a_n}{n^s} = 1 + \sum_{\substack{n>1 \\ \text{only divisible} \\ \text{by } p>P \\ (\Rightarrow n>P)}} \frac{a_n}{n^s} \xrightarrow{P\to\infty} 1$$

$\square$

Cor 3.1.6 $\zeta(s)$ has no zeros with $Re(s) > 1$.

Pf 1 $\zeta(s) = \prod_p \underbrace{\frac{1}{1-\frac{1}{p^s}}}_{\neq 0}$. $\square$

Pf 2 Later...

<u>Lemma 3.1.7</u> A Dirichlet series $D(a,s) = \sum \frac{a_n}{n^s}$ is holomorphic in the region $\{s \in \mathbb{C} : \operatorname{Re}(s) > \sigma_c\}$ with derivative $\frac{d}{ds} D(a,s) = \sum \frac{-a_n \log n}{n^s}$.

<u>Pf</u> The sum is locally uniformly convergent in this region according to Lemma 3.1.2.
Each summand $\frac{a_n}{n^s}$ is holomorphic with derivative $\frac{-a_n \log n}{n^s}$.
That implies the claim. (See e.g. Thm. II.5.2 in Fischer-Lieb: A Course in Complex Analysis)

$\square$

3.2. Meromorphic continuation

Thm 3.2.1 $\zeta(s) = \sum \frac{1}{n^s}$ has a (unique) meromorphic continuation to the entire complex plane, which we will also denote by $\zeta(s)$. Its only singularity is a pole of order 1 and residue 1 at $s=1$:

$\zeta(s) - \frac{1}{s-1}$ is holomorphic everywhere.

Pf Apply Euler-Maclaurin: For all $k \geq 0$ and $\mathrm{Re}(s) > 1$:

$$\sum_{n=2}^{\infty} \frac{1}{n^s} = \underbrace{\int_1^{\infty} \frac{1}{t^s}\, dt}_{= \left[-\frac{1}{s-1} \cdot \frac{1}{t^{s-1}} \right]_{t=1}^{\infty} = \frac{1}{s-1}}$$

$$+ \underbrace{\sum_{r=0}^{k} \frac{(-1)^{r+1}}{(r+1)!} \left[\underbrace{B_{r+1}(t)}_{O(1)} \frac{(-s)\cdots(-s-r+1)}{t^{s+r}} \right]_{t=1}^{\infty}}_{\substack{-B_{r+1}(1)\cdot(-s)\cdots(-s-r+1) \\ \text{(holomorphic in } \mathbb{C})}}$$

$$+ \underbrace{\int_1^{\infty} \frac{(-1)^k}{(k+1)!} \underbrace{B_{k+1}(t)}_{O(1)} \frac{(-s)\cdots(-s-k)}{t^{s+k+1}}\, dt}_{\substack{\text{holomorphic in} \\ \{s \in \mathbb{C} : \mathrm{Re}(s) > -k\}}}$$

$$\Rightarrow \frac{1}{s-1} + \sum_{r=0}^{k} \frac{(-1)^{r+1}}{(r+1)!} \cdot \left(-B_{r+1}(1) \cdot (-s) \cdots (-s-r+1)\right)$$

$$+ \int_1^\infty \frac{(-1)^k}{(k+1)!} B_{k+1}(t) \frac{(-s) \cdots (-s-k)}{t^{s+k+1}} \, dt$$

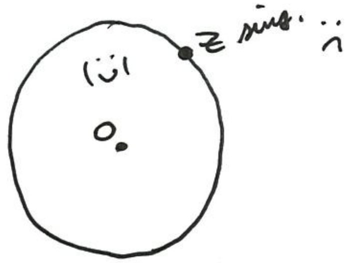
is a meromorphic continuation of $\zeta(s)$ to $\{s \in \mathbb{C} : \operatorname{Re}(s) > -k\}$ for any $k \geq 0$.

(with the claimed singularity only)

$\square$

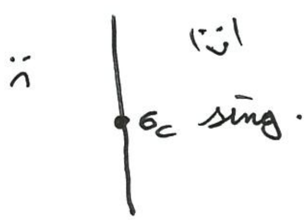
Rmk Power series converge until they cannot due to a singularity:

If $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence r_c, then it has a singularity $z \in \mathbb{C}$ with $|z| = r_c$.



The same holds for Dirichlet series with nonneg. coeff.:

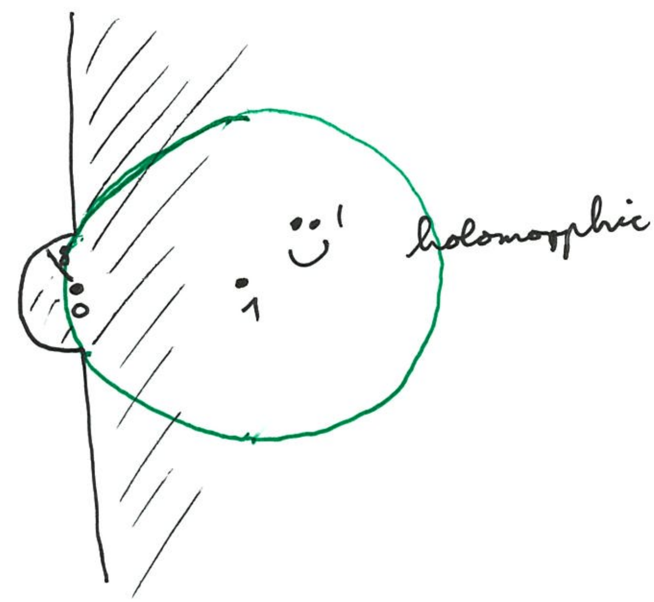
Thm 3.2.2 If $a_1, a_2, \ldots \geq 0$ and $\sigma_c \in \mathbb{R}$, then σ_c is a singularity of $D(a,s)$.



Ex $\zeta(s)$ has a pole at $\sigma_c = 1$.

Pf Replacing a_n by $\frac{a_n}{n^{\sigma_c}}$, we can assume w.l.o.g. that $\sigma_c = 0$.

Assume that $D(s) := D(a, s)$ has a holomorphic continuation to a neighborhood $\{s \in \mathbb{C} : |s| < \delta\}$ of 0.



The Taylor series expansion of $D(s)$ around $s = 1$ is:

$$D(s) = \sum_{k \geq 0} \frac{(s-1)^k}{k!} \cdot D^{(k)}(1) = \sum_{k \geq 0} \frac{(s-1)^k}{k!} \cdot \sum_{n \geq 1} \frac{a_n (-\log n)^k}{n}$$

$$= \sum_{k \geq 0} \sum_{n \geq 1} \frac{(1-s)^k}{k!} \cdot \frac{a_n (\log n)^k}{n}$$

According to the remark, it converges in a circle of radius $\sqrt{1+\delta^2} > 1$.

$\Rightarrow$ It converges at some real $s < 0$.

Since $a_1, a_2, \ldots \geq 0$, each summand is ≥ 0 for $s \leq 0$.

$\Rightarrow$ We can rearrange:

$$\Rightarrow D(s) = \sum_{n \geq 1} a_n \cdot \underbrace{\sum_{k \geq 0} \frac{(1-s)^k}{k!} \cdot \frac{(\log n)^k}{n}}_{\substack{\text{Taylor series for } \frac{1}{n^s} \\ \text{around } s=1}}$$

$$= \sum_{n \geq 1} \frac{a_n}{n^s} \quad \text{converges (for some } s < 0\text{)}.$$

$\Rightarrow \sigma_c < 0.$ $\lightning$ $\square$