Rmk $D(a,s)$ is (formally) invertible if and only if $a_1 \neq 0$.

Pf "$\Rightarrow$" $(a*b)_1 = a_1 b_1$

"$\Leftarrow$" ~~let b(n) := (Σ~~

~~let $b(n) = \sum \frac{1}{a(\cdot)}$~~

Let $b_1 := \frac{1}{a_1}$,

$$b_k := -\frac{1}{a_1} \cdot \sum_{\substack{n,m\geq 1: \\ k=nm \\ m<k}} a_n b_m \qquad \text{(inductively).}$$

$\Rightarrow a*b = \delta.$ $\square$

Def We'll denote the conv. inverse of a sequence a by $\hat{a}$.

Def The Riemann zeta function is

$$\zeta(s) = D(\mathbb{1}, s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } \mathbb{1} = (1,1,\dots).$$

We can use it to make lots of more interesting sequences:

$d = \mathbb{1} * \mathbb{1}$ $\qquad$ $D(d,s) = D(\mathbb{1}*\mathbb{1}, s) = D(\mathbb{1},s)\cdot D(\mathbb{1},s) = \zeta(s)^2$

$d_k = \sum_{\substack{n,m\geq 1: \\ nm=k}} 1\cdot 1 =$ nr. of divisors of k

$d^{(k)} = \underbrace{\mathbb{1} * \dots * \mathbb{1}}_{k \text{ times}}$ $\qquad$ $D(d^{(k)}, s) = \zeta(s)^k$

$id = (1,2,\dots)$ $\qquad$ $D(id, s) = \zeta(s-1)$

$D(id^r, s) = \zeta(s-r)$

$$\sigma = id * \mathbb{1} \qquad\qquad D(\sigma, s) = \zeta(s-1)\,\zeta(s)$$

$$\sigma_k = \sum_{\substack{n,m:\\ k=nm}} n \cdot 1 = \text{sum of divisors of } k$$

$\mu = \widetilde{\mathbb{1}}$ <u>Möbius function</u> $\qquad\qquad D(\mu, s) = \dfrac{1}{\zeta(s)}$

$$\mu_n = \begin{cases} (-1)^k, & n \text{ prod. of } k \text{ distinct primes} \\ 0, & n \text{ not squarefree} \end{cases}$$

$$\varphi * \mathbb{1} = id \qquad\qquad D(\varphi, s) \cdot \zeta(s) = \zeta(s-1)$$

$\varphi_n = \#(\mathbb{Z}/n\mathbb{Z})^\times$ (<u>Euler's phi function</u>)

$= \text{nr. of inv. res. cl. mod } n$

$$(\mathbb{1}_{square})_n = \begin{cases} 1, & n \text{ square} \\ 0, & \text{otherwise} \end{cases} \qquad\qquad D(\mathbb{1}_{square}, s) = \zeta(2s)$$

$\vdots$

Def A sequence $a=(a_1, a_2, \ldots)$ is *multiplicative* if

i) $a_1 = 1$ and

ii) $a_{nm} = a_n a_m$ for all $n,m \geq 1$ with $\gcd(n,m)=1$.

It is *completely multiplicative* if ii) holds for *all* $n,m \geq 1$.

Ex δ, $\mathbb{1}$, id are completely multiplicative.

$\mathbb{1}_{\text{square}}$, d, φ are multiplicative.

Lemma 3.1

a) If a,b are multiplicative, then $a*b$ is.

b) If a is multiplicative and invertible, then $\tilde{a}$ is.

Pf a)
$$(a*b)_{nm} = \sum_{\substack{k,l\geq 1:\\ nm=kl}} a_k b_l \underset{\gcd(n,m)=1}{=} \sum_{\substack{k_1,l_1,k_2,l_2\geq 1\\ n=k_1 l_1\\ m=k_2 l_2}} \underbrace{a_{k_1k_2}}_{a_{k_1}a_{k_2}} \underbrace{b_{l_1l_2}}_{b_{l_1}b_{l_2}}$$
$$= a_n b_m$$

b) similar to a); prove it for $\tilde{a}_n$ by ind. over n. $\square$

Rmk If a is multiplicative, then formally
$$D(a,s) = \prod_{p \text{ prime}} \underbrace{\sum_{k=0}^{\infty} \frac{a_{p^k}}{p^{sk}}}_{1+\frac{a_p}{p^s}+\frac{a_{p^2}}{p^{2s}}+\ldots = F\left((1,a_p,a_{p^2},\ldots), \frac{1}{p^s}\right)}$$
(formal power series in $\frac{1}{p^s}$)

Ex $\zeta(s) = \prod_p \left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\ldots\right) = \prod_p \frac{1}{1-\frac{1}{p^s}}$

Ex $\zeta(s)^2 = D(d,s) = \prod_p \left(1+\frac{2}{p^s}+\frac{3}{p^{2s}}+\frac{4}{p^{3s}}+\dots\right)$

~~Rmk You can formally verify identities. For example:~~

Rule ~~Ex. check whether two~~ To determine a multiplicative sequence a ~~are equal~~, it suffices to know a_{p^k} for all primes p and $k \geq 1$. For example:

Lemma 3.2 Let $\lambda_n = (-1)^{\sum e_i}$ if $n = \prod_i p_i^{e_i}$. Then,

$$\lambda * \mathbb{1} = \mathbb{1}_{square}.$$

Pf 1 Both sides are completely mult.

$$(\lambda * \mathbb{1})_{p^k} = \sum_{\substack{n,m:\\ p^k = nm}} \lambda_n \cdot 1 = \sum_{\substack{t,u \geq 0:\\ k = t+u}} \underbrace{\lambda_{p^t}}_{(-1)^t} \cdot 1 = \begin{cases} 1, & k \text{ even} \\ 0, & k \text{ odd} \end{cases} = \mathbb{1}_{square}(p^k).$$

$\square$

Pf 2

$$D(\lambda * \mathbb{1}, s) = \prod_p \underbrace{\sum_{k \geq 0} \frac{\lambda_{p^k}}{(p^s)^k}}_{f_p(p^s)} \cdot \prod_p \underbrace{\sum_{\ell \geq 0} \frac{\mathbb{1}_{p^\ell}}{(p^s)^\ell}}_{g_p(p^s)} = \prod_p \left(1 + \frac{1}{p^{2s}} + \frac{1}{p^{4s}} + \dots\right) = D(\mathbb{1}_{square}, s)$$

with $f_p(X) = \sum \lambda_{p^k} X^k = 1 - X + X^2 - X^3 \pm \dots = \frac{1-X}{1-X^2}$, $\quad g_p(X) = \sum \mathbb{1}_{p^\ell} X^\ell = 1 + X + X^2 + X^3 + \dots = \frac{1}{1-X}$,

$$f_p(X) \cdot g_p(X) = \frac{1}{1-X^2} = 1 + X^2 + X^4 + \dots$$

$\square$

Rmk $\tilde{\mathbb{1}} = \mu$, where μ is the Möbius function:

$$\mu(n) = \begin{cases} (-1)^k, & \text{if } n \text{ is the product of } k \text{ distinct primes,} \\ 0, & \text{otherwise.} \end{cases}$$

Pf $D(\tilde{\mathbb{1}}, s) = D(\mathbb{1}, s)^{-1} = \prod_{p} (1 - p^{s}) = \sum_{n \geq 1} \frac{\mu(n)}{n^s}$. $\square$

Rmk (Möbius inversion)

If $b_n = \sum_{m \mid n} a_m$ for all $n \geq 1$,

then $a_n = \sum_{m \mid n} b_m \, \mu\left(\frac{n}{m}\right)$ for all $n \geq 1$.

Pf

Assumption $\Leftrightarrow$ $b = a * \mathbb{1}$

Conclusion $\Leftrightarrow$ $a = b * \mu$

$\leftarrow$ $\mu = \tilde{\mathbb{1}}$ is the inverse of $\mathbb{1}$ (w.r.t. convolution)

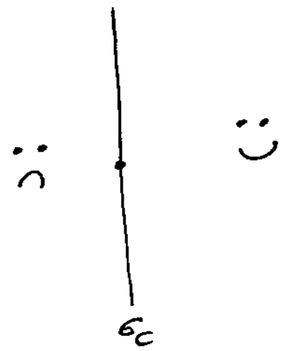
$\square$

3.1. Convergence

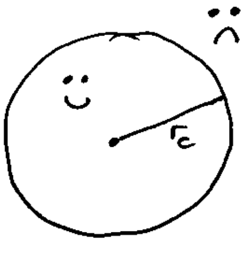
[What does the region of convergence look like?
For power series, it's essentially a disc.
For Dirichlet series, it's essentially a (half-)plane.]

Lemma 3.1.1 Let $s_1, s_2 \in \mathbb{C}$, $\mathrm{Re}(s_1) < \mathrm{Re}(s_2)$.
If $\sum_{n=1}^{\infty} \frac{a_n}{n^{s_1}}$ converges, then $\sum_{n=1}^{\infty} \frac{a_n}{n^{s_2}}$ converges.

Rmk Hence, there is a number $\sigma_c = \sigma_c(a) \in \mathbb{R} \cup \{\pm\infty\}$, called the abscissa of convergence, such that $\sum \frac{a_n}{n^s}$ converges if $\mathrm{Re}(s) > \sigma_c$ and doesn't converge if $\mathrm{Re}(s) < \sigma_c$.



Rmk This is like the radius of convergence for power series.



Pf of Lemma 3.1.1

$$\sum \frac{a_n}{n^{s_1}} \text{ conv.} \Leftrightarrow \sum_{k\le n\le l} \frac{a_n}{n^{s_1}} \xrightarrow[\text{(uniformly in } l)]{k\to\infty} 0$$

$$\sum \frac{a_n}{n^{s_2}} \text{ conv.} \Leftrightarrow \sum_{k\le n\le l} \frac{a_n}{n^{s_2}} \longrightarrow 0$$

Apply Abel summation to $\sum_{k\le n\le x} \frac{a_n}{n^{s_1}}$, $\frac{1}{x^{s_2-s_1}}$ ...

$\square$

Exe $\left(\zeta(s)=\right) \sum_{n=1}^{\infty} \frac{1}{n^s}$ has abscissa of convergence $\sigma_c = 1$.

Pf If $s\in\mathbb{R}$, the sum converges if and only if $s>1$. $\square$

More precisely:

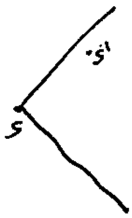
Lemma 3.1.2 ~~for any $\sigma > \sigma_c$, and any $H>0$ $\sum \frac{a_n}{n^s}$ is~~

~~uniformly convergent in~~

~~If~~ If $\sum \frac{a_n}{n^s}$ converges, then $\sum \frac{a_n}{n^{s'}}$ is uniformly convergent in the sector

$\{s'\in\mathbb{C} : \mathrm{Re}(s')\ge \mathrm{Re}(s),\ |\mathrm{Im}(s'-s)|\le H\,\mathrm{Re}(s'-s)\}$

for any $H>0$.



Pf same $\square$