Instead, break up the integral

$$\hat{f'}(a) = \int_{-1}^{1} f'(x) e^{-2\pi i x a} dx \qquad \text{into:}$$

- a piece away from ± 1:

$$\int_{-(1-\frac{1}{a})}^{1-\frac{1}{a}} f'(x) e^{-2\pi i x a} dx \underset{\text{IBP}}{=\!=} \underbrace{\left[f'(x) \cdot \frac{e^{-2\pi i x a}}{-2\pi i a} \right]_{x=-(1-\frac{1}{a})}^{1-\frac{1}{a}}}_{\ll \frac{1}{a^{1/2}}} - \underbrace{\int_{-(1-\frac{1}{a})}^{1-\frac{1}{a}} f''(x) \cdot \underbrace{\frac{e^{-2\pi i x a}}{-2\pi i x a}}_{\ll \frac{1}{a}} dx}_{\ll \frac{1}{a^{1/2}}}$$

- pieces near ± 1:

$$\int_{1-\frac{1}{a}}^{1} \underbrace{f'(x)}_{<0} \underbrace{e^{-2\pi i x a}}_{\ll 1} dx \ll -\left[f(x) \right]_{x=1-\frac{1}{a}}^{1} \ll \frac{1}{a^{1/2}}$$

$$\int_{-1}^{-(1-\frac{1}{a})} \cdots \qquad \ll \frac{1}{a^{1/2}}.$$

$\square$

Thm 2.3.3 (Sierpinski)

$$N(R) = \pi R^2 + \mathcal{O}(R^{2/3}).$$

Rmk Applying Poisson summation to $\mathbb{1}_{B(R)}$ would give an error bound of

$$\sum_{0 \neq t \in \mathbb{Z}^2} \widehat{\mathbb{1}}_{B(R)}(t) \underset{\uparrow}{=} \sum_{0 \neq t \in \mathbb{Z}^2} R^2 \, \widehat{\mathbb{1}}_{B(1)}(Rt)$$

$\mathbb{1}_{B(R)}(x) = \mathbb{1}_{B(1)}\left(\frac{x}{R}\right)$

$$\ll \sum_{0 \neq t \in \mathbb{Z}^2} R^{1/2} |t|^{-3/2} \underset{\uparrow}{=} \infty .$$

$-\frac{3}{2} > -2$

Pf Let $\eta : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be a smooth (radially symmetric) function with $\int_{\mathbb{R}^2} \eta(x)\,dx = 1$ and $\operatorname{supp}(\eta) \subseteq B(1)$.

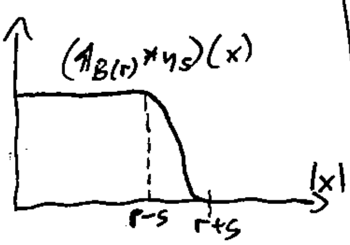
Let $\eta_S(x) = \frac{1}{S^2} \eta\left(\frac{x}{S}\right)$. $(0 < S < R)$

$$\Rightarrow \int_{\mathbb{R}^2} \eta_S(x)\,dx = 1, \quad \hat{\eta}_S(t) = \hat{\eta}(St), \quad \operatorname{supp}(\eta_S) \subseteq B(S)$$

$$\Rightarrow \mathbb{1}_{B(R-S)} * \eta_S \underset{\uparrow}{\leq} \mathbb{1}_{B(R)} \leq \mathbb{1}_{B(R+S)} * \eta_S \qquad (I)$$

$$(\mathbb{1}_{B(R-S)} * \eta_S)(x) = \int_{\mathbb{R}^2} \underbrace{\mathbb{1}_{B(R-S)}(x-y)}_{\leq 1} \eta_S(y)\,dy \leq 1 \;\; \forall x \in B(R)$$

$$\text{—''—} = \underbrace{\int_{\mathbb{R}^2} \underbrace{\mathbb{1}_{B(R-S)}(x-y)}_{\substack{0 \text{ unless} \\ x-y \in B(R-S)}} \underbrace{\eta_S(y)}_{\substack{0 \text{ unless} \\ y \in B(S)}} dy}_{0 \text{ unless } x \in B(R)} = 0 \;\; \forall x \notin B(R)$$



We will later let $s = s(R) \to 0$ as $R \to \infty$.

$$\widehat{(\mathbb{1}_{B(r)} * \eta_s)}(0) = \hat{\mathbb{1}}_{B(r)}(0) \cdot \hat{\eta}_s(0) = \pi r^2 \cdot 1 = \pi r^2$$

$$\sum_{0 \neq t \in \mathbb{Z}^2} \widehat{(\mathbb{1}_{B(r)} * \eta_s)}(t) = \sum_{0 \neq t \in \mathbb{Z}^2} \hat{\mathbb{1}}_{B(r)}(t) \cdot \hat{\eta}_s(t)$$

$$= \sum_{0 \neq t \in \mathbb{Z}^2} r^2 \cdot \underbrace{\hat{\mathbb{1}}_{B(1)}(rt)}_{\ll (r|t|)^{-3/2}} \cdot \underbrace{\hat{\eta}(st)}_{\substack{\ll_k (s|t|)^{-k} \text{ for } k \geq 0 \\ \text{and } \ll 1}}$$

$$\ll_k \sum_{\substack{0 \neq t \in \mathbb{Z}^2: \\ s|t| \geq 1}} r^2 (r|t|)^{-3/2} (s|t|)^{-k}$$

$$+ \sum_{\substack{0 \neq t \in \mathbb{Z}^2: \\ s|t| \leq 1}} r^2 (r|t|)^{-3/2}$$

$$\underset{\text{Lemma}}{\ll} r^{1/2} s^{-k} (s^{-1})^{\frac{1}{2}-k} + r^{1/2} (s^{-1})^{1/2} \qquad \text{for } k \geq 1$$

$$\asymp r^{1/2} s^{-1/2}$$

$$\Rightarrow \sum_{x \in \mathbb{Z}^2} (\mathbb{1}_{B(r)} * \eta_s)(x) = \pi r^2 + O(r^{1/2} s^{-1/2}).$$

With (I): $(S \to 0)$

$$\pi\underbrace{(R-S)^2}_{R^2+O(RS)} + O(R^{1/2}S^{-1/2}) \le N(R) \le \pi\underbrace{(R+S)^2}_{\substack{R^2+O(RS)\\ \uparrow \\ \text{increasing} \\ \text{in } S}} + \underbrace{O(R^{1/2}S^{-1/2})}_{\substack{\text{decreasing}\\ \text{in } S}}$$

$O(RS) + O(R^{1/2}S^{-1/2})$ is smallest (up to bounded factor) when $RS = R^{1/2}S^{-1/2}$, i.e. $S = R^{-1/3}$.

The error term is then $O(R^{2/3})$. $\square$

3. Dirichlet series

In combinatorics, one associates to a sequence $a_0, a_1, \ldots \in \mathbb{C}$ the <u>ordinary generating function</u>

$$F(a,X) = \sum_{n=0}^{\infty} a_n X^n \quad \text{(a formal power series)}$$

Ignoring convergence:

$$F(a,X) + F(b,X) = \sum a_n X^n + \sum b_n X^n = \sum (a_n + b_n) X^n = F(a+b, X)$$

$$F(a,X) \cdot F(b,X) = \left(\sum a_n X^n\right)\left(\sum b_m X^m\right) := \sum_{k=0}^{\infty} \Big(\sum_{\substack{n,m \geq 0: \\ k=n+m}} a_n b_m\Big) X^k = F(a \circledast b, X)$$

$$\frac{d}{dX} F(a,X) = \frac{d}{dX} \sum_{n=0}^{\infty} a_n X^n := \sum_{n=1}^{\infty} n a_n X^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} X^n = F(a', X)$$

$$(a'_n = (n+1) a_{n+1})$$

Rmk These identities hold for any $X \in \mathbb{C}$ for which the LHS is absolutely convergent.

Similarly:
In (multiplicative) number theory, one associates to a sequence $\overset{=a(1)}{a_1}, \overset{=a(2)}{a_2}, \ldots \in \mathbb{C}$ the Dirichlet series

$$D(a,s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s} \qquad \text{(a formal series)}$$

Ignoring convergence:

$$D(a,s) + D(b,s) = D(a+b,s)$$

$$D(a,s) \cdot D(b,s) = \left(\sum \frac{a_n}{n^s}\right)\left(\sum \frac{b_m}{m^s}\right) = \sum_k \Big(\underbrace{\sum_{\substack{n,m\geq 1:\\ k=nm}} a_n b_m}_{(a*b)_k} \Big) \cdot \frac{1}{k^s} = D(a*b, s),$$

where $a * b$ is the number~~-theoretic~~ convolution of a and b

$$\frac{d}{ds} D(a,s) = \sum_{n=1}^{\infty} \frac{-a_n \log n}{n^s} = D(-a \cdot \log, s)$$

pointwise mult.

$$D(a, s-r) = \sum \frac{a_n}{n^{s-r}} = \sum \frac{a_n \cdot n^r}{n^s} = D(a \cdot id^r, s)$$

identity sequence: $id_n = n$

Rmk Again, the identities hold if the LHS is abs. conv.

Rmk The above operations $+, \cdot$ give the set of Dirichlet series (or equivalently, the set of sequences $(+, *$ on$)$) the structure of a ring.

Rmk The mult. identity is $1 = D(\delta, s)$, where $\delta = (1, 0, 0, \ldots)$.