2. Smoothing

2.1. Using Euler-Maclaurin

Root of all evil

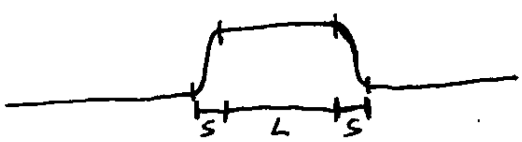
Let I be an interval of length L.

In general, only $\#(I \cap \mathbb{Z}) = L + O(1)$,

not $\#(I \cap \mathbb{Z}) = L$.

Write $\#(I \cap \mathbb{Z}) = \sum_{n \in \mathbb{Z}} \mathbb{1}_I(n)$, where $\mathbb{1}_I$ is the characteristic function of I.

Idea Replace $\mathbb{1}_I$ by a smooth function f.



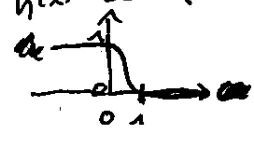
For example, say $I = [0, L]$,

$$f(x) = \begin{cases} 1, & 0 \le x \le 1, \\ \eta\left(\frac{x-1}{s}\right), & 1 \le x, \\ \eta\left(\frac{-x}{s}\right), & x \le 0, \end{cases}$$

where $\eta : \mathbb{R} \to \mathbb{R}$ is a smooth function with

$\eta(x) = 1$ for $x \le 0$

$\eta(x) = 0$ for $x \ge 1$



Thm 2.1.1 We then have

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{\mathbb{R}} f(t)\,dt + O_{\eta,k}(S^{-k})$$ for any η as above and $k \geq 0$.

Pf apply Euler-Maclaurin on an interval $[a,b]$ containing the support of f:

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{\mathbb{R}} f(t)\,dt + \sum_{r=0}^{k} \frac{(-1)^{r+1}}{(r+1)!} \underbrace{\left[B_{r+1}(t) f^{(r)}(t)\right]_{t=a}^{b}}_{0}$$

$$+ \underbrace{\int_a^b \frac{(-1)^k}{(k+1)!} B_{k+1}(t) f^{(k+1)}(t)\,dt}$$

$$\ll_k \int_{\mathbb{R}} |f^{(k+1)}(t)|\,dt$$

$$= 2 S^{-k} \underbrace{\int_{\mathbb{R}} |\eta^{(k+1)}(t)|\,dt}_{< \infty \text{ (indep. of } L, S)}$$

$\square$

2.2. Fourier transforms

Def Let $f \in L^1(\mathbb{R}^n)$ (measurable function $f : \mathbb{R}^n \to \mathbb{C}$ s.t. $\int_{\mathbb{R}^n} |f(x)|\,dx < \infty$)

Its Fourier transform is the function $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ given by:

$$\hat{f}(t) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i (x \cdot t)}\, dx$$

inner (dot) product on $\mathbb{R}^n$

Thm 2.2.1 (Riemann–Lebesgue Lemma)

If $f \in L^1(\mathbb{R}^n)$, then $\hat{f} \in C_0(\mathbb{R}^n)$ (continuous function with $\hat{f}(t) \xrightarrow{|t|\to\infty} 0$)

Ex Let $I = [a,b]$. The Fourier transform of the indicator function 1_I is

$$\hat{1}_I(t) = \int_a^b e^{-2\pi i x t}\, dx = \left[-\frac{1}{2\pi i t} e^{-2\pi i x t} \right]_{x=a}^{b}$$

(If $a = -b$, this is $\frac{1}{\pi t} \sin(2\pi i b t)$.)

__Lemma 2.2.2__ (Basic properties of Fourier transforms)

a) $\hat{f}(0) = \int_{\mathbb{R}} f(x)\,dx$

b) If $g_\lambda(x) = f\left(\frac{x}{\lambda}\right)$ $(\lambda > 0)$, then

$$\hat{g}_\lambda(t) = \lambda^n \cdot \hat{f}(\lambda t).$$

c) Let $n = 1$.
If f is absolutely continuous (f differentiable a.e., f' integrable, $f(b) - f(a) = \int_a^b f'(t)\,dt \ \forall a < b$,
e.g.: continuous and piecewise continuously differentiable),
then $\widehat{f'}(t) = 2\pi i t \cdot \hat{f}(t)$.

__Pf__ a) clear
b) clear
c) integration by parts

$\square$

Thm 2.2.3

If $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n)$, then

$$f(x) = \hat{\hat{f}}(-x) \text{ for } \underbrace{\text{almost all } x \in \mathbb{R}^n}_{\substack{\text{set of bad } x \\ \text{has measure } 0}}.$$

If f is continuous, this holds for all $x \in \mathbb{R}^n$.

Def A smooth function $f : \mathbb{R}^n \to \mathbb{C}$ is a *Schwartz function* if all derivatives of f decay faster than any power of $\frac{1}{|x|}$ for $|x| \to \infty$:

$$|x|^k \left(\frac{\partial}{\partial x_1}\right)^{b_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{b_n} f(x) \xrightarrow{|x|\to\infty} 0 \quad \text{for all } k, b_1, \ldots, b_n \geq 0.$$

The set of Schwartz functions is denoted by $\mathcal{S}(\mathbb{R}^n)$.

Ex Any smooth fct. with compact support.

Rmk
$\mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$
$\mathcal{S}(\mathbb{R}^n) \subseteq C_0(\mathbb{R}^n)$

Thm 2.2.4

If $f \in \mathcal{S}(\mathbb{R}^n)$, then $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$.

(In particular, $|t|^k \hat{f}(t) \xrightarrow{|t|\to\infty} 0$ for all $k \geq 0$.)

Thm 2.2.5 (Poisson summation formula)

If $f \in \mathcal{S}(\mathbb{R}^n)$, then

$$\sum_{x\in\mathbb{Z}^n} f(x) = \sum_{t\in\mathbb{Z}^n} \hat{f}(t).$$

(Note: Both sides are absolutely convergent.)

Rmk $\hat{f}(0) = \int_{\mathbb{R}^n} f(x)\,dx$ is the naive estimate for $\sum_{x\in\mathbb{Z}^n} f(x)$.

So $\sum_{0\neq t\in\mathbb{Z}^n} \hat{f}(t)$ is the error term.

Def The convolution $f*g$ of $f, g \in L^1(\mathbb{R}^n)$ is given by

$$(f*g)(x) = \int_{\mathbb{R}^n} f(x-y)\,g(y)\,dy = \int_{\mathbb{R}^n} f(y)\,g(x-y)\,dy.$$

Lemma 2.2.6 We have $f*g \in L^1(\mathbb{R}^n)$ and

$$\widehat{f*g}\,(t) = \hat{f}(t)\cdot\hat{g}(t) \qquad \forall t \in \mathbb{R}^n.$$

Rmk You can make any $f \in L^1(\mathbb{R}^n)$ smooth by taking the convolution with a smooth fct.

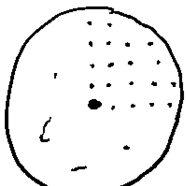
2.3. Gauß circle problem

Goal Estimate $N(R) := \#\{(x,y)\in\mathbb{Z}^2 \mid x^2+y^2\le R^2\}$

$$= \#(\underbrace{B(R)}_{\text{closed ball of radius } R}\cap \mathbb{Z}^2)$$

Rmk The strategy from chapter 1 proves

$$N(R) = \pi R^2 + O(R)$$



Rmk Hardy, Landau showed that $N(R) = \pi R^2 + \Omega(R^{\frac{1}{2}}(\log R)^{\frac{1}{4}})$.

Conjecture $N(R) = \pi R^2 + O_\varepsilon(R^{\frac{1}{2}+\varepsilon}) \quad \forall \varepsilon > 0.$

Known (Huxley) $\quad$ - - - $\; O_\varepsilon(R^{131/208+\varepsilon}) \quad \forall\varepsilon>0.$

We'll show $O(R^{2/3})$.

Lemma 2.3.1 Let $R \geq 0$.

a) $\displaystyle\sum_{\substack{0 \neq x \in \mathbb{Z}^2: \\ |x| \leq R}} |x|^k \sim \frac{2\pi}{k+2} R^{k+2}$ for any real number $k > -2$.

b) $\displaystyle\sum_{\substack{x \in \mathbb{Z}^2: \\ |x| \geq R}} |x|^k \sim \frac{2\pi}{k+2} R^{k+2}$ for any real number $k < -2$.

Pf apply Abel summation to $\underbrace{N(t) - \pi t^2}_{= o_{t\to\infty}(t^2)}$, t^k:

trivial estimate $N(t) \sim \pi t^2$

$$\text{a)}\ \underbrace{-\int_0^R 2\pi t \cdot t^k \, dt}_{\left[\frac{2\pi}{k+2} t^{k+2}\right]_{t=0}^R} + \sum_{\substack{0 \neq x \in \mathbb{Z}^2 \\ |x| \leq R}} |x|^k + \underbrace{\int_0^R \underbrace{(N(t) - \pi t^2)}_{o(t^2)} \cdot k t^{k-1} \, dt}_{o_{R\to\infty}(R^{k+2})} = \underbrace{\left[(N(t) - \pi t^2)\, t^k\right]_{t=0}^R}_{o_{R\to\infty}(R^{k+2})}$$

b) similar

$\square$

Lemma 2.3.2

$$\hat{\mathbb{1}}_{B(1)}(t) \ll |t|^{-3/2} \qquad (\text{for } |t| \to \infty)$$

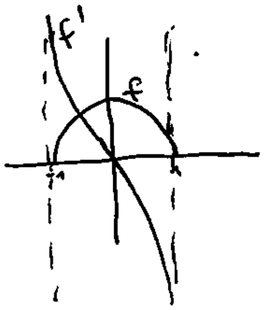
Rmk $\hat{\mathbb{1}}_{B(1)}(t) = \dfrac{J_1(\pi|t|)}{|t|}$, where $J_1(x)$ is the <u>Bessel function</u> of order 1 of the first kind.

Pf By rotational symmetry, we can assume w.l.o.g. that t lies on the positive x-axis: $t = \binom{a}{0}$, $a > 0$.

$$\Rightarrow \hat{\mathbb{1}}_{B(1)}(t) = \int_{B(1)} e^{-2\pi i x a}\, d\binom{x}{y} = 2\int_{\mathbb{R}} f(x) e^{-2\pi i x a}\, dx$$

$$= 2\hat{f}(a)$$

for $f(x) = \begin{cases} \sqrt{1-x^2}, & -1 \le x \le 1, \\ 0, & \text{otherwise}. \end{cases}$



f is abs. cont., so we can apply Lemma 2.2.2 d):

$$\hat{f}(a) = \frac{1}{2\pi i a} \underbrace{\hat{f'}(a)}_{(\ll 1)} \underbrace{\left(\ll \frac{1}{a} = \frac{1}{|t|}\right)}_{\text{not good enough!}}$$

<u>Problem:</u> f' is not abs. cont. near ± 1, so we can't apply Lemma 2.2.2 c) again.