Math 229 – Introduction to Analytic Number Theory

(Fabian Gundlach)

0. Introduction

A few things that can be proved using analysis:

Thm 0.1 (Prime Number Theorem)

$$\#\{p \leq X \text{ prime}\} \underset{X\to\infty}{\sim} \frac{X}{\log X}$$

More precise estimate:

$$\#\{p \leq X \text{ prime}\} \sim \int_2^X \frac{1}{\log t}\, dt.$$

$\rightsquigarrow$ **Heuristic:** The set $\{2, 3, 5, \dots\}$ of prime numbers behaves a little like a random set of natural numbers containing $n \geq 2$ with probability $\frac{1}{\log n}$.

Notation

$f(x) \underset{x\to\infty}{\sim} g(x)$:	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$
$f(x) = o_{x\to\infty}(g(x))$:	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 0$
$f(x) \ll g(x)$ or: $f(x) = O(g(x))$:	$\exists C>0 : \forall x : \lvert f(x)\rvert \le C \cdot g(x)$
$f(k,x) \ll_k g(k,x)$:	$\forall k : \exists C_k>0 : \forall x : \lvert f(k,x)\rvert \le C_k \cdot g(k,x)$
$f(x) \asymp g(x)$:	$f(x) \ll g(x)$ and $f(x) \gg g(x)$
$f(x) = \Omega_{x\to\infty}(g(x))$:	not $f(x) = o_{x\to\infty}(g(x))$ or: $\limsup\limits_{x\to\infty} \frac{\lvert f(x)\rvert}{g(x)} > 0$.

Thm 0.2 (Dirichlet's Thm on primes in arithmetic progressions)

$$\#\{p \le X \text{ prime} : p \equiv a \bmod k\} \sim \frac{1}{\varphi(k)} \cdot \#\{p \le X \text{ prime}\}$$

if a is relatively prime to k, where $\varphi(k) = \#(\mathbb{Z}/k\mathbb{Z})^\times$ is the nr. of residue classes a mod k that are relatively prime to k (invertible). "All invertible res. cl. are equally likely.")

E.g., half the primes are $\equiv 1 \bmod 4$, half are $\equiv 3 \bmod 4$.

Thm 0.3

$$\#\{1 \le n \le X : \exists a, b \in \mathbb{Z} : n = a^2 + b^2\} \sim C \cdot \frac{X}{\sqrt{\log X}} \quad \text{for some } C > 0.$$

Thms 0.1–0.3 are proved using <u>complex analysis</u> (Dirichlet series).

<u>Thm 0.4</u> (special case of Waring's problem) Every positive integer is the sum of at most 19 fourth powers.

This is proved using the circle method.

<u>Thm 0.5</u>

$$\#\{1 \le n \le x \text{ squarefree}\} \sim \frac{6}{\pi^2} \cdot x$$

This is proved using a <u>sieve</u>.

<u>Thm 0.6</u> (Zhang + polymath)
There are infinitely many pairs of primes that differ by ~~exactly 2 (twin prime conjecture)~~ at most 246.

Prerequisites:
- complex analysis
- Fourier analysis
- a little bit of number theory

Grade: 70% weekly homework (probably due Wednesdays) (dropping two lowest scores)

30% take-home final exam

OH this week: Mo, Th 3-4pm in room 233

Course assistant: [redacted] Yujie [redacted] Xu (yujiex@math.harvard.edu)

1. Initiation

1.1. Divisor sum

Def For any integer $n \geq 1$, let $d(n)$ be the number of positive divisors of n:

$$d(n) = \#\{a \mid n\} = \sum_{a \mid n} 1.$$

Ex

n	1	2	3	4	5	6
$d(n)$	1	2	2	3	2	4

Goal Estimate $\sum_{n \leq X} d(n)$ for large X.

Heuristic

$$\sum_{n \leq X} d(n) = \sum_{n \leq X} \sum_{a \mid n} 1 \underset{n = a\cdot b}{=} \sum_{\substack{a,b \geq 1: \\ ab \leq X}} 1$$

$$\overset{(I)}{\approx} \sum_{1 \leq a \leq X} \underbrace{\frac{X}{a}}_{\#\{b:\ ab \leq X\}}$$

$$\overset{(II)}{\approx} \int_1^X \frac{X}{t}\, dt = \left[X \log t\right]_{t=1}^{X} = X \log X.$$

Making (I) rigorous:

$$\#\{1 \leq b \leq \tfrac{X}{a}\} = \left\lfloor \frac{X}{a} \right\rfloor = \frac{X}{a} + O(1),$$

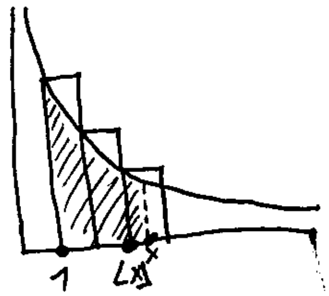
so

$$\sum_{\substack{a,b \geq 1: \\ ab \leq X}} 1 = \sum_{1 \leq a \leq X} \left(\frac{X}{a} + O(1)\right) = \sum_{1 \leq a \leq X} \frac{X}{a} + O(X).$$

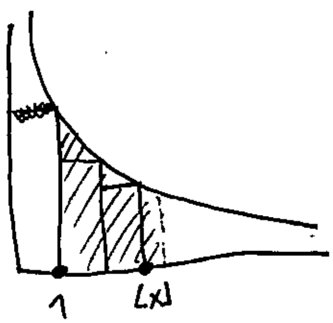
Making (II) rigorous:

Claim $\displaystyle\sum_{1\le n\le X} \frac{1}{n} = \log X + O(1)$ for $X \ge 1$.

Pf



$$\sum_{1\le n\le X} \frac{1}{n} \ge \int_1^X \frac{1}{t}\,dt = \log X$$



$$\sum_{2\le n\le X} \frac{1}{n} \le \int_1^X \frac{1}{t}\,dt = \log X$$

$\square$

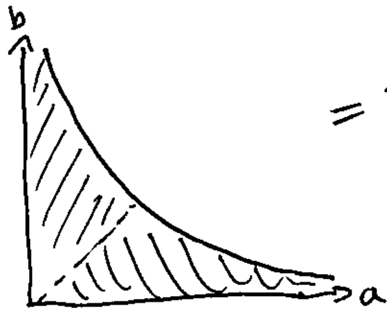
Summary $\displaystyle\sum_{1\le n\le X} d(n) = X\log X + O(X)$,

so the average number of divisors of a random $n \le X$ is $\sim \log X$ for $X \to \infty$.

We can improve the estimate!

Improving (I): ("Dirichlet hyperbola method")

$$\sum_{\substack{a,b\geq 1:\\ ab\leq x}} 1 = \sum_{\substack{a\geq b\geq 1:\\ ab\leq x}} 1 + \sum_{\substack{b\geq a\geq 1:\\ ab\leq x}} 1 - \sum_{\substack{a=b\geq 1:\\ ab\leq x}} 1$$



$$= 2\cdot \sum_{\substack{b\geq a\geq 1:\\ ab\leq x}} 1 - \sum_{\substack{a\geq 1:\\ a^2\leq x}} 1$$

$$= 2\cdot \sum_{1\leq a\leq x^{1/2}} \sum_{a\leq b\leq \frac{x}{a}} 1 - \sum_{1\leq a\leq x^{1/2}} 1$$

$$= 2\cdot \sum_{1\leq a\leq x^{1/2}} \left(\frac{x}{a} - a + O(1)\right) - \left(x^{1/2} + O(1)\right)$$

$$= 2\cdot \sum_{1\leq a\leq x^{1/2}} \frac{x}{a} - x + O(x^{1/2}).$$

better than $O(x)$

1.2. Abel summation

<u>Reminder</u> (Integration by parts)

Let $f, g : [a,b] \to \mathbb{C}$ be continuously differentiable. ($a \le b$)

Then,

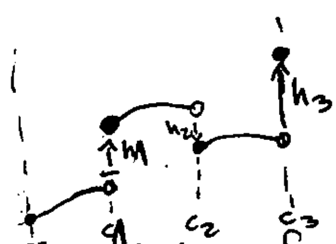
$$\int_a^b f'(t)g(t)\,dt + \int_a^b f(t)g'(t)\,dt = \big[f(t)g(t)\big]_{t=a}^{b} = \big(f(b)g(b) - f(a)g(a)\big).$$

<u>Rmk</u> This continues to hold if f, g are continuous and piecewise continuously differentiable, ignoring points t where $f'(t)$ or $g'(t)$ doesn't exist.
It fails if f, g are not continuous.

<u>Thm 1.2.1</u> (Abel summation)

Let $a = c_0 \le c_1 \le \dots \le c_k = b$, let $f : [a,b] \to \mathbb{C}$ be continuously differentiable on $[c_i, c_{i+1})$ with a jump of height $h_i = f(c_i) - \lim\limits_{\substack{t \nearrow c_i \\ \text{limit from below}}} f(t)$ at c_i

$= \text{"} f(c_i) - f(c_i - \varepsilon) \text{"}$ $(i \ge 1)$



and let $g : [a,b] \to \mathbb{C}$ be continuously differentiable.

Then,

$$\underbrace{\int_a^b f'(t)g(t)\,dt}_{\text{ignore pts } t = c_i} + \sum_{1 \le i \le k} h_i\, g(c_i) + \int_a^b f(t)g'(t)\,dt = \big[f(t)g(t)\big]_{t=a}^{b}.$$