Math 137: Algebraic Geometry

Spring 2022

Problem set #9

due Thursday, April 21 at noon

Throughout, K is assumed to be an algebraically closed field.

Problem 1 (bonus)**.** Let $n \geq 2$ and let $F_d \cong K^{\binom{n+d}{n}}$ be the vector space of polynomials $f \in K[X_1, \ldots, X_n]$ of degree $\leq d$.

a) If $d > 2n - 3$, show that there is a nonempty Zariski open subset $U \subseteq F_d$ such that the set $\mathcal{V}(f) \subseteq K^n$ doesn't contain a straight line for any $f \in U$.

b) If $d < 2n - 3$, show that for every $f \in F_d$, if $\mathcal{V}(f) \subseteq K^n$ contains a straight line, then it contains infinitely many.

c) (too difficult for a bonus problem and totally unfair) If $d \leq 2n - 3$, show that there is a nonempty Zariski open subset $U \subseteq F_d$ such that the set $\mathcal{V}(f)$ contains at least one straight line for all $f \in U$.

Hint: Look at the proof of Theorem 13.5.1. What is the dimension of "the space of straight lines" in K^n? What is the dimension of the space of $f \in F_d$ such that $\mathcal{V}(f)$ contains a particular straight line?

Problem 2. Show that a polynomial $f \in K[X_1, \ldots, X_n]$ vanishes on the entire line spanned by a nonzero vector $x \in K^n$ if and only if all of its homogeneous parts f_d vanish at x.

Problem 3. Let $A = \mathcal{V}(I)$ for an ideal I of $K[X_1, \ldots, X_n]$. Let $S \subseteq K[X_0, \ldots, X_n]$ be the set of homogenizations of elements of I at X_0. Show that $\mathcal{V}_{\mathbb{P}^n_K}(S)$ is the Zariski closure of the image of A under the 0-th standard affine chart map φ_0.

Any invertible linear map $g : K^{n+1} \to K^{n+1}$ induces a map $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ sending the line spanned by $x \in K^{n+1}$ to the line spanned by $g(x) \in K^{n+1}$. Maps $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ of this form are called *projective transformations.*

Problem 4. a) Consider the projective line $\mathbb{P}^1_K = K \sqcup \{\infty\}$. Let P, Q, R be three distinct points in $\mathbb{P}^1_K$. Show that there is a projective transformation $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$ sending P to 0, Q to 1, and R to ∞.

b) We say that points $P_1, \ldots, P_m$ in $\mathbb{P}^n_K$ are *in general linear position* if no $d + 2$ of them lie on a d-dimensional linear subspace for any $0 \leq d \leq \min(m-2, n-1)$.

Let the points $P_1, \ldots, P_{n+2} \in \mathbb{P}^n_K$ be in general linear position and let $Q_1, \ldots, Q_{n+2} \in \mathbb{P}^n_K$ be in general linear position. Show that there is a unique projective transformation $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ sending P_i to Q_i for $i = 1, \ldots, n+2$.

Problem 5 (Pappus's hexagon theorem)**.** Let $g \neq h$ be lines in $\mathbb{P}^2_K$ that intersect in the point P. Let A, B, C be points on g and A', B', C' be points on h (all seven points P, A, B, C, A', B', C' distinct). Let Z be the point of intersection of the lines AB' and $A'B$. Let Y be the point of intersection of the lines AC' and $A'C$. Let X be the point of intersection of the lines BC' and $B'C$. Show that X, Y, Z are colinear. (Hint: Apply a projective transformation to for example make $P = [0 : 0 : 1]$, $A = [1 : 0 : 0]$, $B = [1 : 0 : 1]$, $C = [r : 0 : 1]$, $A' = [0 : 1 : 1]$, $B' = [0 : 1 : 0]$, $C' = [0 : s : 1]$. Then compute X, Y, Z.)