Math 137: Algebraic Geometry

Spring 2022

Problem set #6

due Friday, March 25 at noon

Throughout, K is assumed to be an algebraically closed field.

Problem 1. Show that any algebraic subset of K^n is compact with respect to the Zariski topology. (Every open cover has a finite subcover.)

Problem 2. Let V be an irreducible algebraic set. Show that $\mathcal{O}_V(V) = \Gamma(V)$. (In other words: if a rational function $f \in K(V)$ is defined at every point in V, then $f \in \Gamma(V)$.)

Problem 3. Let V be an irreducible algebraic set, W be any algebraic set, and let $\varphi : V \dashrightarrow W$ be a rational map. We denote by U_φ the set of points $P \in V$ at which φ is defined. Abusing notation, we write $\overline{\varphi(V)} := \overline{\varphi(U_\varphi)}$. We say φ is dominant if $\overline{\varphi(V)} = W$.

a) Show that the map $\varphi : U_\varphi \to W$ is continuous (with respect to the subspace topologies on U_f and W).

b) Show that $\overline{\varphi(V)}$ is irreducible.

c) Let $U \subseteq U_\varphi$ be a nonempty open subset. Show that $\overline{\varphi(U)} = \overline{\varphi(V)}$.

d) Show that φ is dominant if and only if the map $\Gamma(W) \to K(V)$ sending $f \in \Gamma(W)$ to $f \circ \varphi$ is injective.

Problem 4. Are $a = X^2 \in \mathbb{C}(X)$ and $b = X^3 + X + 1 \in \mathbb{C}(X)$ algebraically independent over $\mathbb{C}$? If not, find a polynomial $f \in \mathbb{C}[S, T]$ with $f(a, b) = 0$.

Problem 5. Let I be any ideal of $K[X_1, \dots, X_n]$ and let $V = \mathcal{V}(I)$. Let $S = K[X_1, \dots, X_n]/I$.

a) Show that $\dim(V)$ is the maximum number of (over K) algebraically independent elements of S.
Note: We call elements $f_1, \dots, f_d$ of any K-algebra S *algebraically independent* if there is no polynomial $0 \neq g \in K[Y_1, \dots, Y_d]$ such that $g(f_1, \dots, f_d) = 0$ in S.

b) Show that $\dim(V)$ is the maximum size of a sublist of $X_1, \dots, X_n$ consisting of algebraically independent elements of R.

You can submit the following two problems either with problem set 6 or problem set 7.

Problem 6 (bonus)**.** Consider an ideal I of $K[X_1, \dots, X_n]$ and any monomial order. Show that $\dim(\mathcal{V}(I))$ is the maximal size of a subset $A \subseteq \{1, \dots, n\}$ such that every monomial of the form $\prod_{i \in A} X_i^{e_i}$ is a standard monomial.
Hint: First, think about how to prove this for the degree-lexicographic monomial order. (You might take some inspiration from the next problem.)

Problem 7 (bonus)**.** Let I be an ideal of $K[X_1, \dots, X_n]$. For any $d \geq 0$, let $R_d \subset K[X_1, \dots, X_n]$ be the K-vector space of polynomials of degree at most d and let $\Gamma_d = R_d/(R_d \cap I)$.

a) Show that there is a polynomial $p \in \mathbb{Q}[D]$ and an integer $r \geq 0$ such that for all $d \geq r$, the K-vector space Γ_d has dimension $p(d)$. (This polynomial $p(D)$ is called the *Hilbert polynomial* of I.)
Hint: Use a Gröbner basis and the inclusion–exclusion principle.

b) Show that $\dim(\mathcal{V}(I)) = \deg(p)$.