Math 137: Algebraic Geometry

Spring 2022

Problem set #5

due Wednesday, March 9 at noon

Throughout, K is assumed to be an algebraically closed field.

Problem 1. Let $f \in K[X_1, \dots, X_n]$. Show that if

$$\mathcal{V}\left(f, \frac{\partial f}{\partial X_1}, \dots, \frac{\partial f}{\partial X_n}\right) = \emptyset,$$

then the polynomial f is squarefree.

Problem 2. Show that every monomial order $\leq$ on $\mathcal{S}(X_1, \dots, X_n)$ is a well-order.

Problem 3. Let G be a Gröbner basis of $I \subseteq K[X_1, \dots, X_n]$. Show that the set $\mathcal{V}(I)$ is finite if and only if for all $1 \leq i \leq n$, there is an element $g \in G$ such that $\mathrm{lm}(g) = X_i^t$ for some $t \geq 0$.

Problem 4. a) Let $a, b > 0$ and consider the set

$$V = \{(x, y) \in \mathbb{Z}^2 : x, y \geq 0 \text{ and } ax + by \leq 1\}.$$

Fix any monomial ordering. Show that $X^r Y^s$ is a standard monomial for $\mathcal{I}(V)$ if and only if $(r, s) \in V$.

b) What is the smallest (total) degree d of a nonzero polynomial $f \in \mathcal{I}(V)$?

Problem 5. Let $\leq$ be any monomial order on $K[X_1, \dots, X_n]$ and let $0 \neq f, g \in K[X_1, \dots, X_n]$. Show that if $\gcd(\mathrm{lm}(f), \mathrm{lm}(g)) = 1$, then 0 is a reduction of

$$S(f, g) = \frac{M}{\mathrm{lt}(f)} \cdot f - \frac{M}{\mathrm{lt}(g)} \cdot g$$

with respect to $\{f, g\}$, where $M = \mathrm{lcm}(\mathrm{lm}(f), \mathrm{lm}(g)) = \mathrm{lm}(fg)$.

Problem 6 (bonus)**.** Let $\leq$ be a monomial order on $\mathcal{S}(X_1, \dots, X_n, Y_1, \dots, Y_m)$ such that $A < B$ whenever A is contained in $\mathcal{S}(X_1, \dots, X_n)$ but B isn't contained in $\mathcal{S}(X_1, \dots, X_n)$.

Let G be a Gröbner basis of an ideal $I \subseteq K[X_1, \dots, X_n, Y_1, \dots, Y_m]$.

a) Show that $G \cap K[X_1, \dots, X_n]$ is a Gröbner basis of the ideal $I' = I \cap K[X_1, \dots, X_n]$ of $K[X_1, \dots, X_n]$.

b) Show that the Zariski closure of the image of $\mathcal{V}(I) \subseteq K^{n+m}$ under the projection to K^n (projecting to the first n coordinates) is $\mathcal{V}(I') \subseteq K^n$.