Math 137: Algebraic Geometry

Spring 2022

Problem set #4

due Wednesday, February 25 at noon

Throughout, K is assumed to be an algebraically closed field.

Problem 1. Let $K = \mathbb{C}$ and for any integers $a, b \geq 1$, consider the algebraic subset $V_{a,b} = V(X^b - Y^a)$ of $\mathbb{C}^2$ and the morphism $\varphi_{a,b} : \mathbb{C} \to V_{a,b}$ sending t to (t^a, t^b).

a) For which pairs (a, b) is $\varphi_{a,b}$ injective?

b) For which pairs (a, b) is $\varphi_{a,b}$ surjective?

c) For which pairs (a, b) is $\varphi_{a,b}$ an isomorphism?

d) (bonus) For which pairs (a, b) is $V_{a,b}$ isomorphic to K?

Problem 2. a) Consider the algebraic set

$$V = \{(x, y, z) \in K^3 \mid x^2 + y^2 = z^2\}.$$

Find a nonconstant morphism $\varphi : K \to V$. (Hint: Pythagorean triples.)

b) Consider the algebraic set

$$W = \{(x, y) \in K^2 \mid x^2 + y^2 = 1\}.$$

Assuming that the field K has characteristic zero, show that there is no nonconstant morphism $\psi : K \to W$. (Hint: Pythagorean triples.)

Problem 3. a) Find algebraic subsets V_1, V_2 of $\mathbb{C}^2$ and functions $f_1 \in \Gamma(V_1)$ and $f_2 \in \Gamma(V_2)$ such that $f_1|_{V_1 \cap V_2} = f_2|_{V_1 \cap V_2}$ but there is no function $f \in \Gamma(V_1 \cup V_2)$ with $f|_{V_1} = f_1$ and $f|_{V_2} = f_2$.

b) Corollary 6.2 from class can fail when K is not algebraically closed: Find disjoint algebraic subsets V_1, V_2 of $\mathbb{R}^2$ and functions $f_1 \in \Gamma(V_1)$ and $f_2 \in \Gamma(V_2)$ such that there is no function $f \in \Gamma(V_1 \cup V_2)$ such that $f|_{V_1} = f_1$ and $f|_{V_2} = f_2$.

c) Show that Corollary 6.3 from class still holds when K is not algebraically closed: If $V \subseteq K^n$ is a finite set and $f : V \to K$ any function, there is a polynomial $g \in K[X_1, \dots, X_n]$ such that $f(P) = g(P)$ for all $P \in V$.

Problem 4. Identify the space $M_n(K)$ of $n \times n$-matrices with entries in K with the vector space K^{n^2} (by sending a matrix A to a vector consisting of its entries). For any $r \leq n$, consider the subset $V_r \subseteq M_n(K) = K^{n^2}$ of matrices of rank at most r.

a) Show that V_r is an algebraic subset of K^{n^2}.

b) Show that V_r is an irreducible subset of K^{n^2}. (Hint: Use Problem 7 from Problem Set 3.)