Math 137: Algebraic Geometry

Spring 2022

Problem set #2

due Wednesday, February 9 at noon

Problem 1. Let A be an algebraic subset of K^n and let B be an algebraic subset of K^m. Show that the cartesian product $A \times B$ is an algebraic subset of $K^n \times K^m = K^{n+m}$.

Problem 2. Show that $X = \{(t, e^t) \mid t \in \mathbb{R}\}$ is not an algebraic subset of $\mathbb{R}^2$.

Problem 3. For each of the following ideals I of $\mathbb{C}[X, Y]$, is $1 \in I$? If so, show how to write 1 as a linear combination of the given generators.

a) $I = (X - Y, X^2 + XY - 2Y^2, X + Y - 2)$

b) $I = (X^2 + Y^2 - 1, X + Y - 1, X - Y)$

Problem 4. Let I be an ideal of a polynomial ring $K[X_1, \ldots, X_n]$ over a field K. Let $J = \sqrt{I}$ be its radical. Show that $J^n \subseteq I$ for some $n \geq 1$.

Problem 5. Let K be any field and let A and B be algebraic subsets of K^n. Show that there exists an integer $m \geq n$ and an algebraic subset C of K^m such that the image of C under the projection $K^m \to K^n$ sending $(x_1, \ldots, x_m)$ to $(x_1, \ldots, x_n)$ is the set difference $A \setminus B$.

Problem 6. Give an example of an algebraic field extension L of a field K that is not module-finite (i.e. not a finite-dimensional vector space).

Problem 7. Let K be an infinite field and let $P_1, \ldots, P_m \neq (0, \ldots, 0)$ be m distinct points in K^n. Show that there is an invertible linear map $f : K^n \to K^n$ such that the $n \cdot m$ coordinates of the m points $f(P_1), \ldots, f(P_m)$ are distinct.