Math 137: Algebraic Geometry

Spring 2022

Problem set #10

due Thursday, April 28 at noon

Throughout, K is assumed to be an algebraically closed field.

Problem 1. Let L_1, L_2 be non-intersecting lines in $\mathbb{P}^3$. What is their join (the union of all lines connecting a point on L_1 and a point on L_2)?

Problem 2. Let A be a subset of $\mathbb{P}^n$. Show that the following are equivalent:

i) It is algebraic (meaning $A = \mathcal{V}_{\mathbb{P}^n}(S)$ for some set S of homogeneous polynomials).

ii) The subset $\varphi^{-1}(A)$ of K^n is algebraic for all affine chart maps $\varphi : K^n \to \mathbb{P}^n$.

iii) The subset $\varphi^{-1}(A)$ of K^n is algebraic for all standard affine chart maps $\varphi : K^n \to \mathbb{P}^n$.

Problem 3 (Decomposition into irreducible components)**.** Show that any algebraic subset V of $\mathbb{P}^n$ can be written uniquely as a union $V = V_1 \cup \cdots \cup V_m$ of irreducible algebraic subsets $V_1, \ldots, V_m$ of $\mathbb{P}^n$ with $V_i \subsetneq V_j$ for all $i \neq j$.

Problem 4. Consider the Veronese map $f : \mathbb{P}^1 \to \mathbb{P}^d$ of degree $d \geq 2$, given by $f([x : y]) = [x^d : x^{d-1}y : \cdots : xy^{d-1} : y^d]$.

a) Show that its image is the algebraic set

$$V := \mathcal{V}_{\mathbb{P}^d}(\{Z_k^2 - Z_{k-1}Z_{k+1} \mid 1 \leq k \leq d-1\} \cup \{Z_0 Z_d - Z_1 Z_{d-1}\}).$$

b) Show that there is an inverse morphism $g : V \to \mathbb{P}^1$. (So $f : \mathbb{P}^1 \to V$ is an isomorphism.)

c) Show that there do not exist homogeneous polynomials h_0, h_1 in the variables $Z_0, \ldots, Z_d$ of the same degree e such that $g([z_0 : \cdots : z_d]) = [h_0(z_0, \ldots, z_d) : h_1(z_0, \ldots, z_d)]$ for all $[z_0 : \cdots : z_d] \in V$. (In particular, $h_0(z_0, \ldots, z_d)$ and $h_1(z_0, \ldots, z_d)$ are not simultaneously zero for any $[z_0 : \cdots : z_d] \in V$.)

Problem 5 (bonus)**.** Consider a finite field $\mathbb{F}_q$ of size q.

a) How many points are there in $\mathbb{P}^n_{\mathbb{F}_q}$?

b) For $0 \leq d \leq n$, how many d-dimensional linear subspaces does $\mathbb{P}^n_{\mathbb{F}_q}$ have?

c) For $0 \leq d' \leq d \leq n$ and a d'-dimensional linear subspace L of $\mathbb{P}^n_{\mathbb{F}_q}$, how many d-dimensional linear subspaces M containing L does $\mathbb{P}^n_{\mathbb{F}_q}$ have?