Math 137: Algebraic Geometry

Spring 2022

Problem set #1

due Wednesday, February 2 at noon

Problem 1. Let K be a field and let X be a set of m points in K^n.

a) Show that there is a set $S \subseteq K[X_1, \dots, X_n]$ of size at most n^m such that $X = \mathcal{V}(S)$.

b) Assuming that $K = \mathbb{R}$, show that there is a polynomial $f \in K[X_1, \dots, X_n]$ such that $X = \mathcal{V}(f)$.

c) (bonus) Assuming that the field K is finite, show that there is a polynomial $f \in K[X_1, \dots, X_n]$ such that $X = \mathcal{V}(f)$. (Hint: Use Fermat's little theorem / Lagrange's theorem.)

d) (bonus) Assuming that the field K is infinite, show that there is a set $S \subseteq K[X_1, \dots, X_n]$ of size at most $n+1$ such that $X = \mathcal{V}(S)$.

Problem 2. Show that $A = \{(t, \sin(t)) \mid t \in \mathbb{R}\}$ is not an algebraic subset of $\mathbb{R}^2$.

Problem 3. Consider the *one-sheet hyperboloid*

$$V = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2 + 1\} \subseteq \mathbb{R}^3.$$

Prove that every point $P \in V$ lies on exactly two (straight) lines $l_1, l_2 \subseteq V$.

Problem 4. For every $n \geq 1$, show that the ideal $I = (X, Y)^n$ of $K[X, Y]$ is not generated by n of its elements.

Problem 5. Let K be any field and let A be any subset of K^n. Show that $\mathcal{V}(\mathcal{I}(A))$ is the closure of A with respect to the Zariski topology. (This is called the Zariski closure of A.)