Math 137: Algebraic Geometry

Spring 2022

Final exam

due Friday, May 6 at 11:59pm (ET)

Rules: You may consult the the lecture notes (either those on the course website or those made by students), problem sets, and solutions to problem sets (including the remarks from the graders), and last year's final exam including its solutions. You're not allowed to consult any other references. You may discuss the exam problems only with me (Fabian).

Honor code affirmation: Along with your solutions, please submit the following affirmation:

"I affirm my awareness of the standards of the Harvard College Honor Code."

Turning in the exam: Please submit the exam on Canvas.

You may write your solutions on the computer (e.g. using LaTeX), but clear handwriting is also accepted. If you can't fully solve a problem, still try to write down your ideas!

This exam has six problems. You can get up to 60 points in total. However, I will consider anything greater than or equal to 50 points a full score.

You may cite (without proof) any result from class or from a problem set.

Throughout this exam, K is an algebraically closed field.

Problem 1 (10 points)**.** Let $A \subseteq K^a$, $B \subseteq K^b$, $C \subseteq K^c$ be algebraic subsets and let $\varphi : A \to B$ and $\psi : B \to C$ be morphisms. Assume that the morphism $\psi \circ \varphi : A \to C$ is finite.

a) (3 points) Show that $\varphi : A \to B$ is finite.

b) (4 points) Show that if φ is dominant, then $\psi : B \to C$ is also finite.

c) (3 points) Show that b) can fail without the assumption that φ is dominant.

Problem 2 (10 points)**.** Show that all isomorphisms $\varphi : K \to K$ (of algebraic sets) are of the form $x \mapsto ax + b$ for $a \in K^\times$ and $b \in K$.

Problem 3 (10 points)**.** Let $K = \mathbb{C}$. Show that there is an algebraic subset V of K^n (for some n) and a surjective morphism $\varphi : K^2 \to V$ whose fibers $\varphi^{-1}(P)$ for $P \in V$ are exactly the sets of the form $\{(x, y), (-x, -y)\}$ with $(x, y) \in K^2$.

Problem 4 (10 points)**.** Let $K = \mathbb{C}$. Let

$$H = \{(a, b, c, d) \in K^4 \mid a = c = 0\}$$

and

$$V = \{(a, b, c, d) \in K^4 \mid ab = cd\}$$

and

$$\Delta = \{(P, P) \in K^4 \times K^4 \mid P \in V\}.$$

a) (2 points) What is the dimension of Δ?

b) (4 points) Show that there is no polynomial $f \in K[A, B, C, D]$ such that $H = \mathcal{V}_V(f)$.

c) (4 points) Show that there are no polynomials

$$g_1, g_2, g_3 \in K[A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2]$$

such that $\Delta = \mathcal{V}_{V \times V}(g_1, g_2, g_3)$.

Problem 5 (10 points)**.** Let $n \geq 1$ and $d \geq 0$. Let $F = F_d$ be the vector space of polynomials $f \in K[X_1, \ldots, X_n]$ of total degree at most d. Let A be the set of tuples $(f_1, \ldots, f_{n+1}) \in F \times \cdots \times F$ such that $\mathcal{V}_{K^n}(f_1, \ldots, f_{n+1}) \neq \emptyset$. Show that $\overline{A} \neq F \times \cdots \times F$.

Problem 6 (10 points)**.** Which of the following statements are true? Which are false? (You don't need to give a proof or counterexample.) Any correct answer for a statement gives two points. Any incorrect answer gives zero points. If you don't answer, you get one point.

a) (2 points) Any two birational irreducible algebraic subsets $V \subseteq K^n$ and $W \subseteq K^m$ have the same dimension.

b) (2 points) If $V \subseteq K^n$ and $W \subseteq K^m$ are algebraic sets and $\varphi : V \to W$ is a bijective finite morphism, then φ is an isomorphism.

c) (2 points) If $V \subseteq K^n$ and $W \subseteq K^m$ are algebraic sets, $\varphi : V \to W$ is a morphism, and $P \in \varphi(V)$, then $\dim(V) \leq \dim(W) + \dim(\varphi^{-1}(P))$.

d) (2 points) For every monomial order on the monomials in $X_1, \ldots, X_n$, for all monomials $M < N$, there exists a point $(a_1, \ldots, a_n) \in \mathbb{R}^n$ such that $M(a_1, \ldots, a_n) < N(a_1, \ldots, a_n)$.

e) (2 points) Three planes H_1, H_2, H_3 in $\mathbb{P}^3_K$ (with $H_i \neq H_j$ for all $i \neq j$) always intersect in exactly one point.