Cor 15.5 Identify the vector space F_d of hom. deg. d pol. in $X_0, \dots, X_n$ with $K^{f(d)}$, where $f(d) = \#\{\text{deg. } d \text{ monomials}\} = \binom{n+d}{n}$.

($\Rightarrow$ A point in $\mathbb{P}^{f(d)-1}$ then corresponds to an equivalence class [f] of such polynomials $f \neq 0$ (modulo scaling).)

The points $[f] \in \mathbb{P}^{f(d)-1}$ for which f is reducible form an alg. subset.

Ex Let $d=2$ and $\mathrm{char}(K) \neq 2$. A polynomial $0 \neq f = \sum_{i \leq j} a_{ij} X_i X_j$ is reducible if and only if all 3×3-minors of the matrix

$$M = \begin{bmatrix} a_{00} & \frac{a_{01}}{2} & \frac{a_{02}}{2} & & \\ \frac{a_{01}}{2} & a_{11} & \frac{a_{12}}{2} & & \\ & & a_{22} & & \\ & & & \ddots & \\ & & & & a_{nn} \end{bmatrix}$$

are 0.

(Thanks, Jonas, for pointing out that the original criterion was wrong!) In part., for $n=1$, all f are reducible.

Pf of example Note: M is the matrix representing the bilinear map

$$K^{n+1} \times K^{n+1} \to K$$
$$(v, w) \mapsto \frac{1}{2}\left(f(v+w) - f(v) - f(w)\right)$$

The quadratic form f (and therefore the matrix M) can be diagonalized. This change of coordinates doesn't change whether f is reducible and doesn't change the rank of M.

$\Rightarrow$ Rescaling coordinates, we can assume that

$f = X_0^2 + \ldots + X_{r-1}^2$ (where $1 \le r \le n+1$ is the rank of M,

$$M = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

The polynomial f is clearly reducible if $r=1$ ($f = X_0 \cdot X_0$) or $r=2$ ($f = X_0^2 + X_1^2 = (X_0 + \sqrt{-1}X_1)(X_0 - \sqrt{-1}X_1)$).

If $r \ge 3$, it is irreducible: say $f = gh$ with g, h hom. of deg. 1. $V(g)$ is a hyperplane in K^{n+1}, say spanned by $v_1, \ldots, v_n$. Let $v_1, \ldots, v_n, v_{n+1}$ be a basis of K^{n+1}. If we change to this basis, the matrix M then has the form

$$\begin{bmatrix} 0 & \cdots & 0 & * \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & * \\ * & \cdots & * & * \end{bmatrix},$$

which clearly has rank ≤ 2.

$\square$

Pf of Cor: The set of reducible $[f]$ is the union of the images of the maps $\mathbb{P}^{\epsilon(a)-1} \times \mathbb{P}^{\epsilon(b)-1} \to \mathbb{P}^{\epsilon(d)-1}$ with $a+b=d$, $a, b \ge 1$.

$([g], [h]) \mapsto [gh]$

$\square$

16. Tangent spaces

Def Let V be an alg. subset of K^n. The <u>tangent space</u> to V at $P \in V$ is the vector space

$$T_P(V) := \{ \vec{v} \in K^n \mid D_P(f)(\vec{v}) = 0 \quad \forall f \in J(V) \}$$

where

$$D_P(f) : K^n \longrightarrow K$$
$$(b_1, \dots, b_n) \longmapsto \frac{\partial f}{\partial x_1}(P) \cdot b_1 + \dots + \frac{\partial f}{\partial x_n}(P) \cdot b_n$$

is the derivative of f at P.

Lemma 16.1 If $J(V) = (f_1, \dots, f_m)$, then

$$T_P(V) = \{ \vec{v} \in K^n \mid D_P(f_i)(\vec{v}) = 0 \quad \forall i = 1, \dots, m \}.$$

Pf "$\subseteq$" clear

"$\supseteq$" Let $f = f_1 g_1 + \dots + f_m g_m$.

If $D_P(f_i)(\vec{v}) = 0$ and $f_i(P) = 0$ for all i, then

$$D_P(f)(\vec{v}) = \sum_i \left(D_P(f_i)(\vec{v}) \cdot g_i(P) + f_i(P) \cdot D_P(g_i)(\vec{v}) \right) = 0$$

by the product rule. $\square$

Ese $K=\mathbb{C}$.

$$V=\mathcal{V}(x^2+y^2-1) \Rightarrow \mathcal{I}(V)=(x^2+y^2-1)$$

$$\cancel{\mathcal{I}(V)=(x^2+y^2-1)}$$

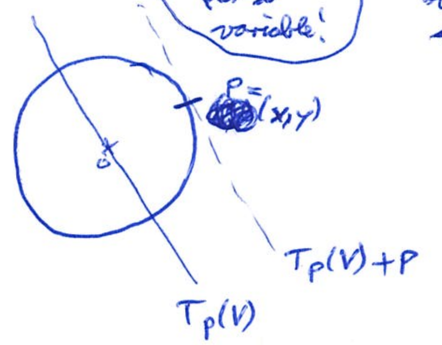
$P=(x,y)\in V$

$$\Rightarrow T_P(V)=\{(r,s)\in K^2 \mid 2xr+2ys=0\}$$

$$=\{(dx,dy)\in K^2 \mid \underbrace{2x\,dx+2y\,dy}_{d(x^2+y^2-1)}=0\},$$

dx is notation for a variable!

the one-dim. vector space "parallel to the line tangent to the circle at (x,y)"



Ese $I=(x^2+y^2-1)^2 \Rightarrow \mathcal{V}(I)=$ same V as before

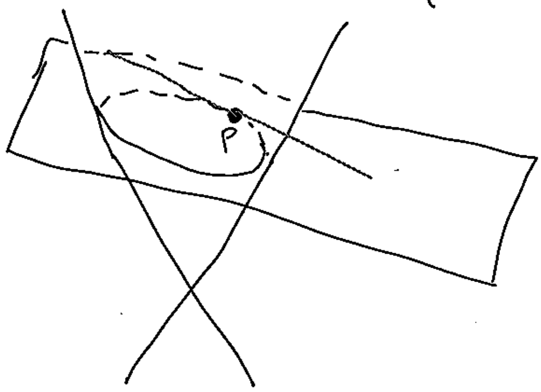
But $T_P(V)=\{(dx,dy)\in K^2 \mid \underbrace{2\underbrace{(x^2+y^2-1)}_{0}(2x\,dx+2y\,dy)}_{d(x^2+y^2-1)^2}=0\}=K^2$.

Ese $K=\mathbb{C}$, $V=\mathcal{V}(x^3-y^2)$, $P=(x,y)$

$$T_P(V)=\{(dx,dy)\in K^2 \mid 3x^2dx=2y\,dy\}$$

$$\dim T_P(V)=\begin{cases}1, & P\neq(0,0),\\ 2, & P=(0,0).\end{cases}$$

Eg $V = V(\underbrace{x^2+y^2-z^2}_{f}, \underbrace{2x+z-11}_{g})$



$P = (x, y, z)$

$(Df)(x,y,z)(a,b,c) = 2xa + 2yb - 2zc$

$\quad \text{---} \ (dx, dy, dz) = 2x\,dx + 2y\,dy - 2z\,dz$

$(Dg)(x,y,z) = (a,b,c) = 2a + c$

$\quad \text{---} \ 2dx + dz$

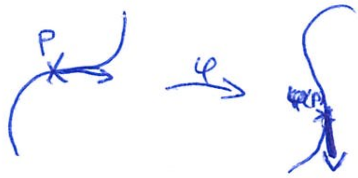
$T_{(x,y,z)}(V) = \ker \begin{pmatrix} 2x & 2y & -2z \\ 2 & 0 & 1 \end{pmatrix}$

$P = (3,4,5) \in V$

$\rightsquigarrow T_P(V) = \left\langle \begin{pmatrix} -4 \\ 13 \\ 8 \end{pmatrix} \right\rangle$.

Rmk If $\varphi: V \to W$ is a morphism, its derivative at $P \in V$ is a map $D_P(\varphi): T_P(V) \longrightarrow T_{\varphi(P)}(W)$.

Pf Let $\vec{v} \in T_P(V)$, $f \in J(W)$.



By the chain rule,

$$D_{\varphi(P)}(f)\left(D_P(\varphi)(\vec{v})\right) = D_P(\underbrace{f\circ\varphi}_{\substack{\in J(V) \\ \text{because} \\ \varphi: V\to W \\ \text{is well-def.}}})(\vec{v}) \underset{\vec{v}\in T_P(V)}{=} 0.$$

□

Rmk In particular, if φ is an isomorphism, then

$$T_P(V) \cong T_{\varphi(P)}(W).$$

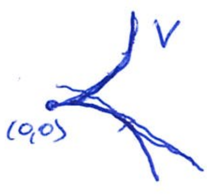
Ex There is no isomorphism $K^n \cong K^m$ (of alg. sets) for $n \neq m$ because the tangent spaces are K^n, K^m, which are nonisomorphic vector spaces.

Ex There is no isom $V = \mathcal{V}(X^3 - Y^2) \cong K$ because V has $T_{(0,0)}(V) = K^2$, but all tangent spaces of K are K^1.

Ex $K = \mathbb{C}$, $V = \mathcal{V}(X^3 - Y^2)$, $P = (x, y)$

$\Rightarrow T_P(V) = \{(dx, dy) \mid 3x^2 dx = 2y\,dy\}$

$$\dim T_P(V) = \begin{cases} 1, & P \neq (0,0), \\ 2, & P = (0,0). \end{cases}$$



Ex $K = \mathbb{C}$, $V = \mathcal{V}(Y^2 - X^2(X+1))$, $P = (x, y)$

$$\dim T_P(V) = \begin{cases} 1, & P \neq (0,0) \\ 2, & P = (0,0) \end{cases}$$

Def Let V be an irreducible alg. set.

V is smooth at P if $\dim T_P(V) = \dim(V)$.

Otherwise, V is singular at P.

V is smooth if it is smooth at every $P \in V$.

Rmk $\dim T_P(V) = \dim\ker\left(\frac{\partial f_i}{\partial x_j}(P)\right)_{i,j}$ if $I(V) = (f_1, \dots, f_m)$.

Rmk For any $d \geq 0$, the set $\{P \in V \mid \dim T_P(V) \geq d\}$ is an algebraic subset of V.

Prop 16.2 Let V be irreducible.

a) $\dim T_P(V) \geq \dim V$ for all $P \in V$.

b) $\dim T_P(V) = \dim V$ for some $P \in V$.

($\Rightarrow$ for all P in a nonempty open subset)

"V is smooth almost everywhere"

Pf See Thm 2.3 in Shafarevich. $\square$

Rmk This is great! For example:

Cor 16.3 Let $P \in V := \mathcal{V}(f_1, \dots, f_m) \subseteq K^n$.

If the matrix $\left(\frac{\partial f_i}{\partial x_j}(P)\right)_{i,j}$ has rank m, then every irred. comp. A of V containing P has codimension m in K^n.

Pf $m \geq \operatorname{codim}(A, K^n) \geq n - \dim T_P(V) = n - \dim\ker\left(\frac{\partial f_i}{\partial x_j}(P)\right)_{i,j} = \operatorname{rk}(\dots) = m$ $\square$

Cor 16.4 Let V be irreducible, smooth at $P \in V$, $\varphi : V \to W$ a dominant morphism. Then, $\dim(W) \geq \operatorname{rk}(D_P(\varphi))$.

Pf ~~Since $\varphi(\varphi^{-1}(\varphi$~~

Let $Q = \varphi(P)$. ~~Since $\varphi(\varphi^{-1}(Q)) = \{Q\}$ and $T_Q(\{Q\}) = 0$, we have~~

~~$T_P(\varphi^{-1}(Q)) \subseteq \ker(D_P(\varphi))$.~~

~~$\Rightarrow \dim$~~

Let A be an irred. comp. of $\varphi^{-1}(Q)$. Since $\varphi(A) = \{Q\}$ and $T_Q(\{Q\}) = 0$, we have

$$T_P(A) \subseteq \ker(D_P(\varphi)).$$

$$\Rightarrow \dim(T_P(A)) \leq \dim \ker(D_P(\varphi)) = \underbrace{\dim(T_P(V))}_{\dim(V)} - \operatorname{rk}(D_P(\varphi))$$

$$\dim(T_P(A)) \geq \dim(A) \geq \dim(V) - \dim(W)$$

$\square$

Rmk a) The assumption that V is smooth at P is necessary: Otherwise, take $\varphi = \mathrm{id}$. $\Rightarrow \operatorname{rk}(D_P(\varphi)) = \dim T_P(V) > \dim(V)$.

b) We can have $>$:

Take $\varphi : K \to K$, $x \mapsto x^2$, $D_0(\varphi) = 0$ has rank 0

c) In char. 0, we have $=$ for some P, but in char. $p > 0$, even that can fail: Take $\varphi : K \to K$, $x \mapsto x^p$ $\Rightarrow D_x(\varphi) = p x^{p-1} = 0$ for all x.